MANY-VALUED LOGICS

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J. B. R. and A. R. T.

I

INTRODUCTION

It has become a truism that every statement is either true or false. It might be supposed that this principle must be disproved before one can write a serious work on many-valued logic. This is by no means the case. In fact, the present volume will not constitute a disproof of such a principle. However, in the following chapters, systems of many-valued logic through the level of the first order predicate calculus will be constructed in such a manner that they are both consistent and complete. The tools of construction will include the logical procedures of ordinary English and those of some formalized systems of two-valued logic. Thus, in effect, we shall use the truism in constructing many-valued logics. It does not follow from this that ordinary two-valued logic is necessary for the construction of many-valued logic, but it does follow that it is sufficient for such constructions. The ability to establish such a sufficiency is certainly no more mysterious than the fact that a Harvard graduate can learn Sanskrit using his native English.

At this point, however, a word of warning is in order, for the treatment of many-valued logic which follows is concerned with the behavior of many-valued statements and not with their meaning. This indifference toward the meaning of many-valued statements indicates that we have no prejudices regarding the possible interpretations of our systems of many-valued logic. As far as our treatment is concerned, the meaning of a many-valued statement could be a linguistic entity such as a many-valued proposition [1] or a physical entity such as one of many positional contacts. [2] For that matter, the meaning of a many-valued statement might be quite different from either a proposition or a positional contact. In any case, regardless of the possible interpretations of many-valued logical systems, it is our opinion that

[1] For example, see Bochvar 1939 and Reichenbach 1944.
[2] For example, see Shannon 1938 and Šéstakov 1946.

most interpretations that have so far been proposed can not be taken too seriously until the precise formal development of such systems has been carried to a level of perfection considerably beyond that which is reached even in the present work. Of course, we do not wish to deny the possibility of finding interpretations for subsystems of many-valued logic such as the statement calculus or predicate calculus of first order, but we do consider various recent proposals for interpretations of many-valued logic definitely premature. [3] Typical of such interpretations are those which concern recent physical theories that involve a theory of measurement requiring the use of numbers. Since a theory of many-valued numbers has not yet been constructed, it is not possible at present to show that the proposed interpretations actually apply for systems of many-valued logic incorporating a theory of numbers. [4] Since the present work does not complete the task of constructing many-valued logics beyond the level of the predicate calculus of first order, if one should judge our results in terms of meaning, then the most that could be claimed for them is that they take us a few steps toward the goal of formalizing many-valued logics which are at least rich enough to include a theory of numbers. In this manner we move a little closer to a decision concerning the applicability of many conjectures regarding interpretations.

As will be seen from the following chapters, the amount of complexity involved in taking our few steps toward the ultimate goal of formalizing many-valued logic is quite considerable, and this might lead the skeptic to question the wisdom of taking such steps, much less any further steps, without considerable assurance of ultimate success in terms of meaning and interpretation. Actually we are willing to take the inevitable risk associated with novel investigations, but in due regard for the skeptic we shall indicate our belief that the gamble has some chance of success. Admittedly, we can not now give conclusive proof of final success, but some favorable evidence can be presented. To this end, consider the following dialogue which might be occasioned by the meeting of

[3] For example, see Reichenbach 1944 and such writings by members of the Destouches group at Paris as Destouches 1942.

[4] For further details concerning the attitude here expressed see Rosser 1941 and Margenau 1950.

certain contemporary logicians who will be referred to as Mr. TURQUER and Mr. ROSSETTE. Mr. Turquer is a spokesman for those twentieth century logicians who believe that two-valued logic embodies the absolute truth, while Mr. Rossette speaks for those who believe that two-valued logic is but a man-made instrument which can be abandoned if the desire or need arises. As might be expected with such representatives, Mr. Turquer defends his case with strong appeals to "truth", "precision", and "accuracy", while Mr. Rossette is an ardent defender of the "rights of men" to use language as they see fit. The entire dialogue, the highlights of which we shall now record, was initiated by only the slightest change in the weather.

Mr. ROSSETTE: It is raining.

Mr. TURQUER: You mean it is raining in Ithaca, New York, at 2 P.M., July 14, 1950, for you do not know whether or not it is now raining in El Paso, Texas.

Mr. ROSSETTE: Would you agree then that my statement is neither true nor false?

Mr. TURQUER: No, that is not my opinion, since every statement is either true or false. Hence, our only conclusion is that what you called a "statement" is really not a statement at all. Rather, it is a statement form or matrix implicitly containing free variables of time, place, etc., which must be bound in order to convert the matrix into a statement.

Mr. ROSSETTE: But is this not a bit arbitrary? It seems to me that you assume that every statement is either true or false and then distinguish between statements and statement forms in order to escape being refuted. Why not assume that statements are either true or false or neither? Surely this would be just as acceptable as your distinction between statement and statement form.

Mr. TURQUER: Your proposal is equally arbitrary, for it amounts te defining some alternative things which you choose to call *statements* and which are either statements or statement forms. You then argue that if P is a statement from the class of *statements*, then P is true or false according as the statement P is true or false, while if P is a statement form from the class of

statements, then P is neither true nor false. This most certainly does not deny the fact that statements remain either true or false.

Mr. ROSSETTE: I would prefer to say that one can not deny an interesting fact about Mr. Turquer, namely, that he is determined to distinguish between statements and statement forms. If I choose to so decide, I could argue that neither is the fact altered that *statements* are either true or false or neither.

Mr. TURQUER: Well, if the situation is as arbitrary as all that, then I shall merely remark that you are voicing the early opinion of Hugh MacColl, [5] and in such arbitrary matters it is well to let history choose between the alternatives. You realize, of course, that history has not decided in favor of Mr. MacColl's views.

Mr. ROSSETTE: I do not grant that the decisions of history are always sound. Only a few generations ago, the decision of history was overwhelmingly in favor of the divine right of kings and a universe based on Euclidean geometry. Now both doctrines are quite in disfavor. However, for the sake of argument I will grant your distinction between statements and statement forms. Nevertheless, I believe that there are some actual statements which are possibly neither true nor false.

Mr. TURQUER: You know you are being ridiculous. I challenge you to produce such a statement.

Mr. ROSSETTE: Well, suppose that when the janitor arrives we ask him if he is in this room. No doubt, he will reply in the affirmative and with complete assurance, but if the question is repeated as he leaves and while he is passing through the door, then he will certainly be flustered and unable to give an answer.

Mr. TURQUER: Oh, that is no problem, for all we need to do is inform the janitor of the necessity of specifying some boundaries to the room which will define precisely when he is in and out of the room.

Mr. ROSSETTE: But what about *him*? Will it not also be necessary to specify the physical boundaries of the janitor's body?

[5] For example, see MacColl 1896.

Mr. TURQUER: Yes indeed, or we might just as well use the center of gravity of the janitor's body, specifying that when this is beyond a certain point, the janitor's body is out of the room. In the last analysis, all such difficulties are easily resolved by indicating the presence of certain free variables which must be bound before statement forms can become statements.

Mr. ROSSETTE: If I accept your solution, then I am forced to the conclusion that the janitor's original assurance about being in the room was completely illusory, for at that time he was as ignorant of the precise meaning of being in and out of the room as he was when we repeated the question at the time he was leaving through the door. Actually, from your point of view, he could only utter statement forms until he became versed in the use of free and bound variables. Thus, it would appear that ordinary discourse must consist entirely of statement forms. If so, it must fail completely ever to convey meaning. But after all, scientific discourse developed from ordinary language.

Mr. TURQUER: I would rather say that scientific discourse divorced itself from the vague expressions and ambiguities of ordinary language. In a nutshell, it learned the importance of distinguishing statements from statement froms and, in general, it learned the necessity of precise definition.

Mr. ROSSETTE: I doubt that the transition from ordinary common sense to scientific knowledge is as sharp as you would have it, [6] but again for the sake of argument, let us suppose that it is and further examine the question of the truth or falsity of all statements.

Mr. TURQUER: By all means let us get on with a scientific discussion. Ordinary discourse is just so much nonsense to me.

Mr. ROSSETTE: That is another interesting fact about you, but to be scientific let us return to the center of gravity of our janitor. We would like to specify the precise center of gravity of the janitor's body in order to be able to define exactly when he is in or out of the room. How should this be done in an acceptable scientific manner?

[6] See Born 1949.

Mr. Turquer: Perhaps the best method would be to specify the collection of atoms which constitute the janitor's body, and using their positions calculate the exact center of gravity of this body.

Mr. Rossette: But the janitor's body is at a temperature which can be estimated, so we can estimate the thermal velocities of the atoms which compose his body. However, does not the principle of indeterminacy assert that if we have any information whatever of the velocities of atoms, then exact information concerning their positions is impossible? If so, we could not give an exact definition of the center of gravity of our poor janitor's body.

Mr. Turquer: Your point is cogent only if we fail to recognize that the statements of quantum physics are probability statements. It most certainly is possible to be exact about the numerical probability that the center of gravity of the body in question is at a certain point, and the probability will be sufficiently high to give us a location for the center of gravity which is quite accurate enough for practical purposes in the macrocosmic world of our janitor's body.

Mr. Rossette: This still leaves me in the air concerning the exact point at which the macrocosm ends and the microcosm begins, not to mention the fact that one would like an exact definition of "accurate enough for practical purposes", if your position is correct.

Mr. Turquer: Such questions as you ask can be generated without end, but like ordinary expressions, the fact that they can be framed does not guarantee that they make any sense.

Mr. Rossette: That's a neat trick of evasion.

Mr. Turquer: You know very well that it would not be difficult to define the boundary line between the microscosm and macrocosm if we wished to take the time, and the same goes for "practical accuracy". A certain choice for a precise numerical probability would turn the trick in either case.

Mr. Rossette: I am not so sure, for a numerical statement of probability is merely a statement to the effect that if a certain

kind of experiment is repeated more and more times tending toward infinity, then the frequency of a certain type of occurrence approaches a certain limit. Clearly, this is not a testable affair. Hence, we have no convincing evidence that the corresponding statement of an exact numerical probability is either true or false.

Mr. TURQUER: Very well then. You have displayed a statement such that neither the laws of physics nor the laws of probability prove that the statement must be either true or false. However, there are still the laws of logic, and they assure us that the statement must indeed be either true or false.

Mr. ROSSETTE: Do you mean the laws of logic, or the history and tradition of logic?

Mr. TURQUER: I don't get you.

Mr. ROSSETTE: Let's look, then, at the more general case and put the point like this: I agree that there has developed a tradition, and a glorious one, of two-valued logic. This may have been the result of its simplicity and very considerable scientific success from at least the time of Euclid up to the present day. Your "laws of logic" are just part of this tradition.

Mr. TURQUER: Such a long and glorious history is strong evidence of the truth.

Mr. ROSSETTE: Am I to conclude, then, that you are a believer in the divine right of kings and in a Euclidean construction of the universe?

Mr. TURQUER: Not at all. We have learned the limitations of both kings and Euclidean geometries but not of two-valued logic.

Mr. ROSSETTE: But in time we may so learn.

Mr. TURQUER: I do not admit that there are any limitations of two-valued logic to be learned.

Mr. ROSSETTE: You *are* a stickler for tradition! Don't you believe in bold experiments, and aren't you curious about the results of experimenting with alternative logics?

Mr. TURQUER: I fear that such experiments are only the result of an idle curiosity.

Mr. ROSSETTE: I can not agree. After all, there are many-valued logics in which two-valued logic can be embedded [7] and surely such many-valued logics would work as well as ordinary two-valued logic.

Mr. TURQUER: You forget that the truth and simplicity go hand in hand. Why use a more complex logic when a simple logic will do as well?

Mr. ROSSETTE: But I question whether it will do as well. Might not the alternative between the traditional two-valued logic and many-valued logic be analogous to that between the once traditional Euclidean geometry and the now more physically acceptable non-Euclidean geometry?

Mr. TURQUER: Perhaps, but I have little confidence in arguments by analogy.

Mr. ROSSETTE: There is the further point that highly exact sciences such as modern physics have long since learned the value of having possible models with a wealth of formal structures and inter-relationships, for these greatly increase the chances of finding useable models. We already feel fairly confident that many-valued logic offers a much greater wealth of structure than the ordinary two-valued logic.

Mr. TURQUER: I will be happier about many-valued logics after the applications are found, and from all the effort which is required just to construct these logics, there would appear to be little hope for finding applications any time soon. Hence, I would prefer to spend my time sharpening well established tools.

Mr. ROSSETTE: Since we can find no a priori reason against it, I think I would prefer to help those who are interested in performing a bold experiment.

At this point the dialogue was ended by the entrance of the janitor whose center of gravity had proved so elusive, and who now announced that it was time to clean the room which Mr. Rossette and Mr. Turquer had been occupying during their debate.

[7] For example, see Hoo 1949 and Słupecki 1946.

Hence, no proof which was mentioned in the dialogue was ever completed. Yet, we feel that enough evidence was presented to make it unreasonable to attempt to persuade Mr. Rossette not to undertake his bold experiment. On the other hand, we sympathize with the claim of Mr. Turquer that such an experiment is not likely to yield any definite results within the near future, and that our final evaluation of many-valued logics must rest with just those results of the distant future. However, if all men practiced the extreme caution of a Mr. Turquer, no scientific progress would be possible. No doubt it would be equally disastrous to revolt blindly against tradition as some have done, but careful self-conscious experimentation is not of the nature of a blind revolt.

It is in the spirit of just such cautious and self-conscious experimentation that the present work on many-valued logics is written. We wish to advance a little further the bold experimental work of men like Mr. Rossette. It is the purpose of the remainder of this volume to report our findings, but we wish to emphasize now that our results go only as far as giving a general solution to the problem of constructing and formalizing many-valued predicate calculi of first order. Even this advances the formal theory of many-valued logics, and the success with which these results have been achieved gives us added faith not only in the possibility of developing a formal theory of many-valued logics which is at least as highly refined as that which has been achieved in ordinary two-valued logic, but also in the possibility of finding useful applications for many-valued logical systems. However, though our faith has been increased, it is not to be concluded that we are prejudiced about the final experimental results. On the contrary, we are committed to the thesis that in the practice of a scientific discipline one must withhold final judgment until all the experimental data has been reported. With this in mind, we shall turn first to the study of many-valued statement calculi, and then using this as a foundation, we shall develop a general theory of many-valued predicate calculi of first order. In all of this, we do not presuppose any previous knowledge of many-valued logics on the reader's part; however, he should be well acquainted with the technical manipulations of the ordinary two-valued predicate calculus of first order.

II

TRUTH TABLES

Ever since there was first a clear enunciation of the principle "Every proposition is either true or false", there have been those who questioned it. With the development of an axiomatic treatment of logic, it has become possible to construct systems of logic in which this principle is not valid. One way to obtain a usable set of axioms for scuh a purpose is to replace this principle by an alternative one such as "Every statement is true or false or tertium".

One's first reaction to this might be that henceforth we no longer have the principle of reductio ad absurdum. Certainly, we no longer have it in the familiar form of a "tertium non datur". Instead, we have a generalized form which may be called a "quartum non datur".

If such a principle is used, it is clear that we are operating with *three truth* values, namely "truth", "falsehood", and "tertium". We trust the reader will forgive us for using "tertium" both as an adjective and a related noun, but adequate vocabulary to deal with this situation is lacking.

Without dwelling further on these and other peculiarities of a system with three truth values, let us go directly to the general case of M truth values. For this purpose we shall list the following hypotheses, letting Hi denote the ith hypothesis:

H 1. A certain finite or denumerably infinite set of symbols is given. A formula is considered to be any finite linear succession of symbols from the given set of symbols, where a symbol may occur more than once in a given formula. Then a certain non-empty subset of all formulas is chosen as the set of statements for our given logic. Such statements are often defined recursively but for our purposes this is not essential. We will commonly write P, Q, R, S, etc., with or without subscripts, to denote individual (unspecified) statements.

It will be noted that H 1 assures us of the existence of a state-

ment P which is such that there is no statement with a smaller number of symbols than P. Henceforth we will refer to such a statement P as "a least statement" of our given set of statements.

H 2. We will fix attention on b functions of statements which shall be denoted as follows:

$$
\begin{aligned}
&F_1(P_1, \ldots, P_{a_1}) \\
&F_2(P_1, \ldots, P_{a_2}) \\
&\cdot \\
&\cdot \\
&\cdot \\
&F_b(P_1, \ldots, P_{a_b})
\end{aligned}
$$

where $b \geqslant 1$ and $a_i \geqslant 1$ for $1 \leqslant i \leqslant b$. That is, if $P_1, \ldots, P_{a_i}$ are statements, then $F_i(P_1, \ldots, P_{a_i})$ is a statement.

Familiar examples of such statement functions from the two-valued statement calculus are $\sim P$, $P \supset Q$, and $P \,\&\, Q$. When we say that attention is fixed on b functions, we mean that these functions are given initially and, hence, constitute at least some of the functions of our given logic. By combining these, as in $(\sim P) \supset Q$, one can in general construct many more statement functions.

H 3. If Q is one of $P_1, \ldots, P_{a_i}$, then the number of symbols in Q is less than the number of symbols in $F_i(P_1, \ldots, P_{a_i})$ and each $P_j (1 \leqslant j \leqslant a_i)$ occurs in $F_i(P_1, \ldots, P_{a_i})$.

It is clear from H 3 that we are assured of the existence of infinitely many statements.

H 4. If $F_i(P_1, \ldots, P_{a_i})$ is the same statement as $F_j(Q_1, \ldots, Q_{a_j})$, then $i = j$ and P_1 is the same statement as Q_1, P_2 is the same statement as $Q_2, \ldots, P_{a_i}$ is the same statement as Q_{a_j}.

It will be observed that in H 1 to H 4 nothing has been said about truth values. In fact, these hypotheses only define the substructure for our statement calculus. However, we will now list some hypotheses which involve the use of truth values.

H 5. M is an integer such that $M \geqslant 2$, and we refer to the positive integers $1, \ldots, M$ as "truth values".

H 6. S is a truth value such that $1 \leqslant S < M$, and we refer to the truth values $1, \ldots, S$ as "designated". The truth values $S + 1, \ldots, M$ are referred to as "undesignated".

Note that 1 is always designated and M is always undesignated.

We are here proceeding in analogy with the two-valued case. Of the two usual truth values, one is attached to statements which are asserted and one to statements which are denied. Likewise, in a many-valued logic there will be statements which are asserted, and statements which are denied. An asserted statement will be characterized by having a designated truth value attached to it, while a statement that is denied will be characterized by having an undesignated truth value attached. If one wishes a bit of motivation, then one can imagine that the truth values indicate the degree of assertability of a statement to which they are attached, with the lesser truth values denoting greater assertability. The truth value S is then the dividing line between the barely assertable and the not quite assertable.

H 7. With each $F_i(P_1, \ldots, P_{a_i})$ there is associated a truth-value function $f_i(p_1, \ldots, p_{a_i})$ such that if $p_1, \ldots, p_{a_i}$ are positive integers between 1 and M inclusive, then so is the value of $f_i(p_1, \ldots, p_{a_i})$.

It is common to express this assumption by saying that if the truth values $p_1, \ldots, p_{a_i}$ are assigned to the statements $P_1, \ldots, P_{a_i}$, then the value of $f_i(p_1, \ldots, p_{a_i})$ is the truth value of $F_i(P_1, \ldots, P_{a_i})$. More briefly, if q denotes the value of $f_i(p_1, \ldots, p_{a_i})$, then it is often said that $F_i(P_1, \ldots, P_{a_i})$ takes the truth value q when the truth values $p_1, \ldots, p_{a_i}$ are assigned to the statements $P_1, \ldots, P_{a_i}$ respectively. In what follows use will be made of these common expressions, but in such cases it is to be understood that reference is being made to H 7.

If attention is confined to truth-value relationships, then we need never concern ourselves with the statements or the functions $F_i(P_1, \ldots, P_{a_i})$, but attention may be restricted entirely to the truth-value functions $f_i(p_1, \ldots, p_{a_i})$ and these may be defined by some set of truth tables. In such a case, many-valued statement calculi are characterized by a choice of M, S, and $f_i(p_1, \ldots, p_{a_i})$. The choice of S does not affect the internal structure of the calculus,

but only indicates which of the statements are to be considered assertable. However, this is a very relevant consideration, as we shall see later when we undertake to set up axiom systems for many-valued statement calculi.

Since P is the identity function of P, there is actually one more function available than the list of $F_i(P_1, \ldots, P_{a_i})$ indicates. In H 7 we could require that the identity function P have associated with it a truth-value function $f(p)$. However, we shall always assign the identity function p to the identity function P. Hence, this identity function must be included as well as the $f_i(p_1, \ldots, p_{a_i})$ in any complete listing of the basic truth functions. It may happen that the identity function p can be formed by combining the $f_i(p_1, \ldots, p_{a_i})$ even in cases when the identity function P can not be constructed out of the $F_i(P_1, \ldots, P_{a_i})$. For example, in the familiar two-valued statement calculus the statement $\sim\sim P$ is not identical with the statement P, but the truth function corresponding to both $\sim\sim P$ and P is the identity truth function. Note also that as a result of H 3, we know that the identity truth function is the only possible truth function which is to be associated with a least statement.

Actually, the same statement calculus may be determined by two quite different sets of basic truth functions. This comes about because one can use not only the basic $f_i(p_1, \ldots, p_{a_i})$, but also any function obtained by combining these in any manner we like, using any or all of them, and as often as is desired. It is this totality of constructible functions, together with a specification of M and S, which determines a many-valued statement calculus. If this totality of constructible functions comprises all possible truth functions of the truth values $1, \ldots, M$, then we say that the calculus is "functionally complete". If a calculus is not functionally complete, then we call it "functionally incomplete". Cases of such completeness and incompleteness will be illustrated in what follows.

We have purposely avoided any reference to the logical attributes which might be attached to our statements. Indeed, one of the advantages of our treatment is that it relies entirely on the purely formal attributes of statements. From this point of view, in order to have precision one must define a "statement formula of the

statements $P_1, \ldots, P_n$". To this end, let $P_1, \ldots, P_n$ be distinct statements, and choose $p_1, \ldots, p_n$ as distinct truth-value variables; that is the values of $p_i (1 \leqslant i \leqslant n)$ are positive integers less than or equal to M. We now give our definition by induction as follows:

(α) $P_i (1 \leqslant i \leqslant n)$ is a statement formula of the statements $P_1, \ldots, P_n$ and p_i is its corresponding truth-value function.

(β) If $Q_1, \ldots, Q_{a_i}$ are statement formulas of the statements $P_1, \ldots, P_n$ and $q_1, \ldots, q_{a_i}$ are their corresponding truth-value functions respectively, then $F_i(Q_1, \ldots, Q_{a_i})$ is a statement formula of the statements $P_1, \ldots, P_n$ and $f_i(q_1, \ldots, q_{a_i})$ is its corresponding truth-value function.

It is important to observe that a statement can be considered as a statement formula in more than one way, and as a consequence may have more than one corresponding truth function. Thus, let $P \supset Q$ be a statement formula and $\supset (p, q)$ its corresponding truth function. We may then consider $P \supset (Q \supset R)$ as a statement formula of P_1, P_2, and P_3 where P_1 is P, P_2 is Q, and P_3 is R. Its corresponding truth function will be $\supset (p_1, \supset (p_2, p_3))$. Alternatively, $P \supset (Q \supset R)$ may be thought of as a statement formula of P_1 and P_2 where P_1 is P, and P_2 is $Q \supset R$. Its corresponding truth function will then be $\supset (p_1, p_2)$. Finally, $P \supset (Q \supset R)$ may be considered as a statement formula of P_1 where P_1 is $P \supset (Q \supset R)$. In this case its corresponding truth function is the identity function p_1.

It should be remarked that what we are here calling a "statement formula of the statements $P_1, \ldots, P_n$" is sometimes referred to as a "statement (statement formula) which is built up out of the statements $P_1, \ldots, P_n$". Hence, in what follows we shall feel free to use this latter kind of expression as a synonym for our well-defined expression "statement formula of the statements $P_1, \ldots, P_n$".

In accordance with our notion that assertable statements shall take one of the designated truth values $1, \ldots, S$, and only assertable statements shall have designated truth values, we wish to arrange our definition of assertable statements in such a manner as to bring about the following result:

If Q is a statement formula built up out of $P_1, \ldots, P_n$ and $q(p_1, \ldots, p_n)$ is its corresponding truth function, then Q is an assertable statement in case $q(p_1, \ldots, p_n)$ is such that its value is

designated no matter what set of truth values is assigned to the p's.

One way to bring about this desired result is to take it as part of the definition of "assertable statement". In the next chapter we shall disclose some schemes for writing out sets of axioms which will also bring about the desired result. However, regardless of the method used to achieve the specified result, such a method must assure us that a statement will be assertable in case there is some way to consider it as a statement formula for which its corresponding truth function takes only designated values.

So far, the statement formulas and their corresponding truth functions have been quite unspecified. It is natural to ask if one can define statement formulas in many-valued logic which are analogous to those in two-valued logic which are usually translated as "and", "or", "If — then —", etc. That these definitions are possible, may be seen by considering such many-valued operators as the Łukasiewicz–Tarski[8] K, A, and C which are commonly translated as "and", "or", and "conditional" respectively. If $P_1 K P_2$, $P_1 A P_2$, and $P_1 C P_2$ are statement formulas built up out of P_1 and P_2 using K, A, and C, then their respective corresponding truth functions are $\max(p_1, p_2)$, $\min(p_1, p_2)$, and $\max(1, p_2 - p_1 + 1)$. Clearly for $M = 2$ and $S = 1$, these become the truth functions which correspond to the statement formulas $P_1 \,\&\, P_2$, $P_1 \vee P_2$, and $P_1 \supset P_2$ of ordinary two-valued logic. Also, for $M > 2$ these truth functions are such that a lesser truth value corresponds to greater assertability as in the ordinary two-valued case. For example, $P K Q$ is just as assertable as the least assertable of P and Q; $P A Q$ is just as assertable as the most assertable of P and Q; and $P C Q$ has the greatest assertability when Q is at least as assertable as P. Finally, the truth functions corresponding to $P K Q$ and $P A Q$ satisfy the familiar commutative, associative, and distributive laws as in the ordinary two-valued logic.

It should be remarked that we have departed from the Polish notation of Łukasiewicz and Tarski in which statement formulas are not built up using K, A, and C in the same manner that statement formulas of two-valued logic are built up using &, $\vee$, and $\supset$. In general, we have constructed statement formulas after the

[8] Łukasiewicz–Tarski 1930.

pattern of ordinary two-valued logic for the benefit of those who are not acquainted with the Polish notation. Clearly, our discussion is not altered essentially by such a change in notation. If the Polish notation had been adopted, we would write KPQ, APQ, and CPQ in the place of $P\,K\,Q$, $P\,A\,Q$, and $P\,C\,Q$. In the same way we could write $\&PQ$, $\vee PQ$, and $\supset PQ$ rather than $P \,\&\, Q$, $P \vee Q$, and $P \supset Q$. The great advantage of Polish notation is to be found in the fact that it allows one to avoid the use of parentheses. Henceforth, we will use the Polish notation only in cases where systems of Polish logic such as that of Łukasiewicz and Tarski are our special objects of study. In such cases, however, we shall so frame our definitions that the use of Polish notation should offer no difficulty even to those who are not familiar with this particular brand of symbolism.

For the purpose of negating statements, Łukasiewicz and Tarski introduced an operator N which is such that the truth function corresponding to NP_1 is $M - p_1 + 1$. It turns out that N plays a role in many-valued logic which is similar to the role played by $\sim$ in such two-valued principles as $(P \,\&\, Q) \equiv \sim (\sim P \vee \sim Q)$, $\sim\sim P \equiv P$, and $(P \supset Q) \equiv (\sim Q \supset \sim P)$. This may be checked by observing that $P\,K\,Q$ (or KPQ) and $N((NP)\,A\,(NQ))$ (or $NANPNQ$) have the same corresponding truth function, that NNP and P have the same corresponding truth function, and that $P\,C\,Q$ (or CPQ) and $(NQ)\,C\,(NP)$ (or $CNQNP$) have the same corresponding truth function.

There is another role played by $\sim$ in two-valued logic which can not be played by N in many-valued logic. In two-valued logic the operator $\sim$ is such that $\sim P$ is assertable when and only when P takes the truth-value "false". Note that this would not be the case for NP and P if $M = 3$ and $S = 2$ in which case the truth value "false" is to be identified with the truth value 3. In order to have many-valued operators that will play this important role which $\sim$ plays in two-valued logic, we shall introduce the operators $J_k(-)$ $(1 \leqslant k \leqslant M)$ which are such that in effect $J_k(P)$ is assertable when and only when P takes the truth value k. That is, if $j_k(p)$ is the truth function corresponding to $J_k(P)$, then the value of $j_k(p)$ is designated when and only when the value of p is k. We thus have several functions, $J_{S+1}(P)$, $J_{S+2}(P)$, $\ldots$, $J_M(P)$

playing a role that was played by $\sim$ in two-valued logic. That a single function sufficed for this role in two-valued logic is due to the circumstance that in two-valued logic there could be only a single undesignated value. Actually in a similar fashion, $J_1(P)$, $J_2(P), \ldots, J_S(P)$ play a role that is played by the identity function P in the two-valued case. It can now be seen that $J_1(P)\ A\ J_2(P)\ A\ \ldots\ A\ J_M(P)$ is a many-valued generalization of the two-valued principle $P \vee \sim P$.

Another role of $\sim$ is that it converts an assertable statement into an unassertable statement and vice versa. We can now define a many-valued operator $\overline{}$ which will behave analogously. For this, define $\overline{P}$ as $J_{S+1}(P)\ A\ J_{S+2}(P)\ A\ \ldots\ A\ J_M(P)$. Clearly $\overline{}$ converts assertable statements into unassertable statements and vice versa. Note also that $P\ A\ \overline{P}$ is always assertable just as in two-valued logic. Still other generalizations of the two-valued $\sim$ have been proposed.[9]

Actually, for most purposes the Łukasiewicz–Tarski C is not a suitable generalization of the two-valued $\supset$. For instance, in general we do not have a principle of modus ponens in the form, "If P and $P\ C\ Q$ are assertable, then Q is assertable". To see this consider any logic in which $M > 2$ and $S > 1$, and notice that max $(1, (S+1) - S + 1) = 2$. However, it is not difficult to define a many-valued operator for which our principle of modus ponens holds. For this purpose define $P \supset Q$ as $\overline{P}\ A\ Q$ and we see that if P and $P \supset Q$ are both assertable then so is Q.

There are many-valued operators which have no counterpart in two-valued logic. The Słupecki[10] operator T is a case in point. This Słupecki operator is such that the truth function corresponding to any statement formula TP always takes the value 2, i.e., the corresponding truth function has the constant value 2. Later on we shall have occasion to study the operator T in some detail.

As a specific instance of our general argument, suppose that in H 2 we choose $b = 2$, $a_1 = 2$, $a_2 = 1$, and let $F_1(P_1, P_2)$ and $F_2(P_1)$ be the Łukasiewicz–Tarski $P\ C\ Q$ (or CPQ) and NP. Then in H 7, $f_1(p_1, p_2)$ is $\max(1, p_2 - p_1 + 1)$ and $f_2(p_1)$ is $M - p_1 + 1$. By

[9] For example, see Post 1921, Bochvar 1939, Rosser 1941, and Reichenbach 1944.

[10] Słupecki 1936.

combining these basic statement functions many others can be defined with various corresponding truth functions. Thus, if we define $P \vee Q$ as $F_1(F_1(P, Q), Q)$ and $P \,\&\, Q$ as $F_2(F_2(P) \vee F_2(Q))$, then their corresponding truth functions $\vee(p_1, p_2)$ and $\&(p_1, p_2)$ will be $\min(p_1, p_2)$ and $\max(p_1, p_2)$ respectively. Hence, their truth functions are the same as those that correspond to $P\ A\ Q$ and $P\ K\ Q$ respectively, and we can let $P \vee Q$ and $P \,\&\, Q$ as thus defined play the role of our many-valued "or" and "and".

Perhaps more surprising is the fact that we can also define statement functions $J_k(P)$ with the corresponding truth functions $j_k(p)$ having the following truth-value properties:

$$j_k(p) = \begin{cases} 1 & \text{if } p = k \\ M & \text{if } p \neq k \end{cases}$$

Note that this $j_k(p)$ is such that it takes a designated value when and only when $p = k$. It thus satisfies the truth-value properties desired of the truth function corresponding to $J_k(P)$. This follows in view of H 6. We will now give a recursive definition of $J_k(P)$ which consists of a set of inductive rules of procedure for constructing $J_k(P)$ for any given M and S.[11]

We first define a statement function $H_t(P)$ as follows:

(α) $H_1(P) =_{\mathrm{df}} F_2(P)$

(β) $H_{t+1}(P) =_{\mathrm{df}} F_1(P, H_t(P))$

Let $h_t(p)$ be the truth function corresponding to $H_t(P)$. We will now establish the truth-value properties of $h_t(p)$.

Theorem[12] 2. 1. $h_t(p) = \max(1, M - (p - 1)t)$.

Proof. Use induction on t.

(α) If $t = 1$, then $h_t(p) = f_2(p) = M - p + 1 = M - (p - 1)t$. Since by H 7, $1 \leqslant p \leqslant M$, $M - (p - 1)t \geqslant 1$. Hence, $h_t(p) = \max(1, M - (p - 1)t)$.

(β) Assume the theorem for t and prove it for $t + 1$. If we let v temporarily denote $h_{t+1}(p)$, then $v = f_1(p, h_t(p)) = \max(1, h_t(p) - p + 1)$. Now consider the following cases:

Case 1. If $h_t(p) = 1$, then $h_t(p) - p + 1 = 2 - p$. Thus, since

[11] For another definition see Hempel 1937.

[12] In numbering our theorems we will let "*Theorem* $k.l$" denote the lth theorem of chapter k.

$1 \leqslant p \leqslant M$, $v = \max(1, h_t(p) - p + 1) = \max(1, 2 - p) = 1$. But by assumption (β), $h_t(p) = \max(1, M - (p-1)t) = 1$. Hence, $M - (p-1)t \leqslant 1$ and $M - (p-1)(t+1) < 1$. Since $v = 1$, we have $v = \max(1, M - (p-1)(t+1))$.

Case 2. If $h_t(p) > 1$, then by (β) we have $h_t(p) = \max(1, M - (p-1)t) = M - (p-1)t$. Hence, $h_t(p) - p + 1 = h_t(p) - - (p-1) = M - (p-1)t - (p-1) = M - (p-1)(t+1)$. Therefore, $v = \max(1, h_t(p) - p + 1) = \max(1, M - (p-1)(t+1))$.

This completes the proof of our theorem, and we now define $J_1(P)$ as follows:

$$J_1(P) =_{\mathrm{df}} F_2(H_{M-1}(P))$$

We will show that by this definition $j_1(p)$ has the desired truth-value properties.

Theorem 2. 2. If $p = 1$, then $j_1(p) = 1$ and if $p \neq 1$, then $j_1(p) = M$.

Proof. By theorem 2. 1, $h_{M-1}(p) = \max(1, M - (p-1)(M-1))$. Hence, if $p = 1$, then $h_{M-1}(p) = M$ and $j_1(p) = f_2(h_{M-1}(p)) = = M - M + 1 = 1$. But if $p \neq 1$, then $h_{M-1}(p) = 1$ and $j_1(p) = = M - 1 + 1 = M$.

Next we will wish to define $J_k(P)$ for $1 < k < M$, but before this is possible it is necessary to define a particular integer which we will temporarily denote as I.

Let I be defined as the least integer such that $I \geqslant (M-k)/(k-1)$ where $1 < k < M$.

Since $k \geqslant 2$, it is clear that I exists, and since $k < M$, $I \geqslant 1$. Before we can give our definition of $J_k(P)$ for $1 < k < M$, however, it will be necessary to prove some lemmas involving the use of I.

Lemma[13] 2. 3. 1. $(M-k)/(k-1) \leqslant I < (M-1)/(k-1)$.

Proof. If $I \geqslant (M-1)/(k-1)$, then $I - 1 \geqslant (M-k)/(k-1)$ which contradicts the definition of I as the least integer such that $I \geqslant (M-k)/(k-1)$.

Lemma 2. 3. 2. $h_I(k) = M - (k-1)I$.

Proof. By lemma 2. 3. 1, we have $M - 1 > (k-1)I$. Hence, $M - (k-1)I > 1$. So by theorem 2. 1, $h_I(k) = M - (k-1)I$.

[13] In numbering our lemmas, we shall let "*Lemma k.l.m.*" denote the mth lemma used in proving the lth theorem of chapter k.

Lemma 2. 3. 3. $k \geqslant h_I(k) > 1$.

Proof. To show that $h_I(k) > 1$ use the proof of lemma 2. 3. 2. By lemma 2. 3. 1, $M - k \leqslant (k-1)I$ and $M - (k-1)I \leqslant k$. Hence, by lemma 2. 3. 2 we have $h_I(k) \leqslant k$.

Lemma 2. 3. 4. If $p < k$, then $h_I(p) > h_I(k)$.

Proof. By lemma 2. 3. 2 and 2. 3. 3, $h_I(k) = M - (k-1)I > 1$. Since $p < k$, $M - (p-1)I > M - (k-1)I$ and by theorem 2. 1 we have $h_I(p) > h_I(k)$.

Lemma 2. 3. 5. If $p > k$, then $h_I(p) < h_I(k)$.

Proof. As in the proof of lemma 2. 3. 4, $h_I(k) = M - (k-1)I > 1$. Since $p > k$, $M - (p-1)I < M - (k-1)I$. Also, since $M - (k - 1)I > 1$, $h_I(k) > \max(1, M - (p-1)I)$. Thus, by theorem 2. 1 we have $h_I(p) < h_I(k)$.

We are now in a position to define $J_k(P)$ for $1 < k < M$. Induction will be used in that we assume that each of $J_1(P), \ldots, J_{k-1}(P)$ has already been defined before we attempt to define $J_k(P)$. The definition will be by cases as determined in lemma 2. 3. 3.

Case 1. If $k = h_I(k)$, then use the following definition:

$$J_k(P) =_{\text{df}} J_1(F_1(H_I(P) \vee P,\ H_I(P) \,\&\, P))$$

Case 2. If $k > h_I(k)$, then use the following definition where r temporarily denotes $h_I(k)$:

$$J_k(P) =_{\text{df}} J_r(H_I(P))$$

In applying these definitions we will proceed according to the following plan:

Step 1. Define $J_1(P)$ as indicated.

Step 2. Define $J_2(P)$ as indicated in case 1. This is required since $k = 2$, and by lemma 2. 3. 2, $h_I(k) = h_I(2) = M - (2-1)I = M - I$. But I is the least integer such that $I \geqslant (M-2)/(2-1)$. That is, $I = M - 2$. Hence, $h_I(k) = M - (M-2) = 2 = k$.

Step 3. Define $J_3(P)$ as indicated in case 1 if $3 = h_I(3)$, but if $3 > h_I(3)$ use case 2 and step 2. This latter alternative is required since by lemma 2. 3. 3, $h_I(3) > 1$, and since $h_I(3) < 3$, we have $h_I(3) = 2$. Hence, $J_3(P)$ is $J_2(H_I(P))$.

Step 4. Define $J_4(P)$ as indicated in case 1 if $4 = h_I(4)$, but if $4 > h_I(4)$ use case 2 and step r where $r = h_I(4)$.

Step 5. Define $J_5(P)$ as in case 1 if $5 = h_I(5)$, but if $5 > h_I(5)$ use case 2 and step r where $r = h_I(5)$.

. . . .

Step $M-1$. Define $J_{M-1}(P)$ as in case 1 if $M-1 = h_I(M-1)$, but if $M-1 > h_I(M-1)$ use case 2 and step r where $r = h_I(M-1)$.

Later we shall define $J_M(P)$ and, thus, indicate how to take step M. For the moment, however, it will be shown that if $J_k(P)$ is defined according to our plan, then for each k ($k < M$) the corresponding truth function $j_k(p)$ has the truth-value properties stated in the next theorem.

Theorem 2. 3. If $p = k$ $(1 \leqslant k < M)$, then $j_k(p) = 1$ and if $p \neq k$, then $j_k(p) = M$.

Proof. Use induction on k.

(α) If $k = 1$, then $J_k(P)$ is defined by step 1 and our theorem follows by theorem 2. 2.

(β) Assume the theorem for each step before the kth and consider $J_k(P)$ as defined by step k. We then have the following cases:

Case 1. If $k = h_I(k)$, then consider the following subcases:

Subcase 1. If $p=k$, then $p=h_I(k)=h_I(p)$. Hence, $p=\mathsf{v}(h_I(p), p) = \&(h_I(p), p)$, and $f_1(\mathsf{v}(h_I(p), p), \&(h_I(p), p))=f_1(p, p)=\max(1, p-p+1) = 1$. Therefore, $j_k(p) = j_1(1) = 1$ by theorem 2. 2.

Subcase 2. If $p < k$, then by lemma 2. 3. 4 we have $h_I(p) > h_I(k)$. Hence, $h_I(p) > k$ and $p < h_I(p)$. This means that $f_1(\mathsf{v}(h_I(p), p), \&(h_I(p), p)) = f_1(p, h_I(p)) = \max(1, h_I(p) - p + 1 > 1$. Therefore, by theorem 2. 2, $j_k(p) = j_1(f_1(\mathsf{v}(h_I(p), p), \&(h_I(p), p))) = M$.

Subcase 3. If $p > k$, then proceed as in the proof of subcase 2 using lemma 2. 3. 5 in the place of lemma 2. 3. 4.

Case 2. When $k > h_I(k)$, consider the following subcases:

Subcase 1. If $p = k$, then $h_I(p) = h_I(k)$, and $j_k(p) = j_r(h_I(p)) = j_r(r)$. But $r < k$ and so by our assumption (β) we have $j_r(r) = 1$ and $j_k(p) = 1$.

Subcase 2. If $p < k$, then by lemma 2. 3. 4, $h_I(p) > h_I(k)$. Thus, $j_k(p) = j_r(h_I(p)) = M$ by assumption (β), and the fact that $h_I(p) \neq r$.

Subcase 3. If $p > k$, then proceed as in the proof of subcase 2 using lemma 2. 3. 5 in the place of lemma 2. 3. 4.

To obtain step M define $J_M(P)$ as follows:

$$J_M(P) =_{\text{df}} J_1(F_2(P))$$

Theorem 2. 4. If $p = M$, then $j_M(p) = 1$ and if $p \neq M$, then $j_M(p) = M$.

Proof. If $p = M$, then $f_2(p) = f_2(M) = M - M + 1 = 1$. Hence, by theorem 2. 2, $j_M(p) = j_1(f_2(p)) = j_1(1) = 1$. If $p \neq M$, then by H 7, $p < M$. Hence, $f_2(p) = M - p + 1 > 1$, and by theorem 2. 2 we have $j_M(p) = j_1(f_2(p)) = M$.

With the proof of theorem 2. 4, it is clear that we have solved the problem set for ourselves earlier. That is, $J_k(P)$ $(1 \leqslant k \leqslant M)$ can be so defined in terms of F_1 and F_2 (or C and N) that for each k the corresponding truth function $j_k(p)$ has the required truth-value properties.

It is now possible to define $\overline{P}$ as $J_{S+1}(P) \vee \ldots \vee J_M(P)$, and $P \supset Q$ as $\overline{P} \vee Q$. If $\overline{}(p)$ and $\supset(p, q)$ denote the truth functions which correspond to $\overline{P}$ and $P \supset Q$ respectively, then it is clear that these truth functions have the following truth-value properties which are generalizations of the ordinary two-valued "not" and "conditional":

$$-(p) = \begin{cases} 1 & \text{if } p \text{ is undesignated} \\ M & \text{if } p \text{ is designated} \end{cases}$$

$$\supset(p, q) = \begin{cases} 1 & \text{if } p \text{ is undesignated} \\ q & \text{if } p \text{ is designated} \end{cases}$$

It follows that in terms of our choice for the basic functions $F_1(P_1, P_2)$ and $F_2(P_1)$, one can define generalizations of any of the familiar functions of the ordinary two-valued logic. Nevertheless, if $M > 2$, then a many-valued logic based on our F_1 and F_2 is functionally incomplete. That is, a many-valued logic based on the Łukasiewicz–Tarski operators C and N is not functionally complete [14]. This results from the fact that any statement function which is built up using only F_1 and F_2 is such that its corresponding truth function $f(p_1, \ldots, p_n)$ always takes the value 1 or M if only the values 1 or M are assigned to $p_1, \ldots, p_n$. Thus, there will be

[14] Słupecki 1936.

no statement whose corresponding truth function is 2 for all values of $p_1, \ldots, p_n$.

However, if in addition to F_1 and F_2 we allow the use of an $F_3(P_1)$ which is taken to be the Słupecki "tertium function" TP_1 already mentioned, then we have a truth function $t(p_1) = f_3(p_1)$ which is always 2, and it can be shown that a many-valued logic based on F_1, F_2, and F_3 is functionally complete for any choice of M and S. In order to prove this, it will be necessary to establish some preliminary results. First, it will be convenient to introduce the following definitions:

Definition of $\sum_1^n P_i$:

(α) $\sum_1^1 P_i =_{\text{df}} P_1$

(β) $\sum_1^{n+1} P_i =_{\text{df}} (\sum_1^n P_i) \vee P_{n+1}$.

Definition [15] of $\Phi_w(P)$:

(1) If $w = 1$, $\Phi_w(P) =_{\text{df}} F_1(P, P)$.

(2) If $1 < w < M$, $\Phi_w(P) =_{\text{df}} F_1(H_w(F_3(P)), F_2(F_3(P)))$.

(3) If $w = M$, $\Phi_w(P) =_{\text{df}} F_2(F_1(P, P))$.

If $\phi_w(p)$ denotes the truth function corresponding to $\Phi_w(P)$, then the following lemma establishes the truth-value properties of $\phi_w(p)$.

Lemma 2. 5. 1. For every p, $\phi_w(p) = w$.

Proof. There are three cases to consider, but when $w = 1$ or $w = M$ the theorem is obvious by the definition of $\phi_w(p)$. Consider the case where $1 < w < M$. $\phi_w(p)$ is then $f_1(h_w(f_3(p)), f_2(f_3(p))) =$ $= f_1(h_w(2), f_2(2))$. But $f_2(2) = M - 1$, and by theorem 2. 1, $h_w(2) =$ $= \max(1, M - w)$. Since $w < M$, $h_w(2) = M - w$. Hence, $\phi_w(p) =$ $= f_1(M-w, M-1) = \max(1, (M-1)-(M-w) + 1) = \max(1, w) = w$ since $w > 1$.

Now consider any truth function of one argument, and let $g(i)$ $(1 \leqslant i \leqslant M)$ denote its value when i is the value of its argument. We will show how to construct a statement formula (using the basic functions F_1, F_2, F_3) whose corresponding truth function is equal to g. To this end let $\Psi_g(P)$ be defined as follows: $\Psi_g(P) =_{\text{df}} \sum_{i=1}^{M}(\Phi_{g(i)}(P) \,\&\, J_i(P))$ where g is the chosen truth function of one argument. If $\psi_g(p)$ denotes the truth function which cor-

[15] This definition is temporary in the sense that it is used only in the present proof of functional completeness.

responds to $\Psi_g(P)$, we wish to show that $\psi_g(p) = g(p)$ and, hence, that $\Psi_g(P)$ is a statement formula whose corresponding truth function is equal to the chosen truth function of one argument g. The purpose of the next lemma is to prove that $\Psi_g(P)$ has this latter property.

Lemma 2. 5. 2. If $1 \leqslant k \leqslant M$, then $\psi_g(k) = g(k)$.

Proof. Let z temporarily denote a chosen $g(k)$. By the definition of $\Psi_g(P)$, the kth term in the defining sum is $\Phi_z(P)$ & $J_k(P)$. Let $f_k(p)$ denote the truth function corresponding to this kth term. Then $f_k(p) = \&(\phi_z(p), j_k(p)) = \max(\phi_z(p), j_k(p))$. If $p = k$, then by lemma 2. 5. 1 and the truth-value properties of $j_k(p)$ we have $f_k(p) = \max(z, 1) = z = g(k)$. Now each term of the defining sum in the definition of $\Psi_g(P)$ except the kth is of the form $\Phi_{g(i)}(P)$ & $J_i(P)$ where $i \neq k$. Let $f_i(p)$ denote the truth function corresponding to such a term. Then, as above, $f_i(p) = \max(\phi_{g(i)}(p), j_i(p))$. Hence, if $p = k$, $f_i(p) = \max(g(i), M) = M$ since $i \neq k$. But by definition, $\psi_g(k) = \min(M, \ldots, f_k(k), \ldots, M) = \min(M, g(k)) = g(k)$.

Let $g(p_1, \ldots, p_n)$ be a truth function of n arguments. Note that there are M^{M^n} distinct truth functions of n arguments. We wish to show that given any one of these distinct truth functions $g(p_1, \ldots, p_n)$, a statement formula $F(P_1, \ldots, P_n)$ built up out of $P_1, \ldots, P_n$ using our basic F_1, F_2, and F_3 can be constructed in such a way that its corresponding truth function $f(p_1, \ldots, p_n) = g(p_1, \ldots, p_n)$. That is, we wish to show that our logic based on F_1, F_2, and F_3 (C, N, and T) is functionally complete. This is the import of the next theorem.

Theorem 2. 5. If $g(p_1, \ldots, p_n)$ is one of M^{M^n} distinct truth functions that are possible, then there is a statement formula $G(P_1, \ldots, P_n)$ built up out of $P_1, \ldots, P_n$ using F_1, F_2, and F_3 such that its corresponding truth function is $g(p_1, \ldots, p_n)$.

Proof. Use induction on the number of arguments in $g(p_1, \ldots, p_n)$.

(a) If $n = 1$, there are M^M distinct $g(p_1)$'s. These $g(p_1)$'s will be distinguished by their different sets of values as p_1 takes values from 1 to M. Choose any one of these distinct $g(p_1)$'s and call it g. By lemma 2. 5. 2 and the definition of $\Psi_g(P)$, we can take $\Psi_g(P_1)$ as our statement formula $G(P_1)$ and its corresponding truth function $\psi_g(p_1) = g(p_1)$.

(β) Assume the theorem for all distinct truth functions of n arguments and prove it for all the distinct truth functions with $n+1$ arguments.

Let $g(p_1, \ldots, p_{n+1})$ be one of the distinct truth functions with $n+1$ arguments. By our assumption (β), there is a statement formula $G_i(P_2, \ldots, P_{n+1})$ whose corresponding truth function $g_i(p_2, \ldots, p_{n+1}) = g(i, p_2, \ldots, p_{n+1})$ where i is a constant such that $1 \leqslant i \leqslant M$. Hence, we can construct a statement formula defined as follows:

$$G(P_1, \ldots, P_{n+1}) =_{\mathrm{df}} \sum_{1}^{M} (J_i(P_1) \mathbin{\&} G_i(P_2, \ldots, P_{n+1})).$$

If $g^*(p_1, \ldots, p_{n+1})$ denotes the truth function corresponding to $G(P_1, \ldots, P_{n+1})$ and $p_1 = i$ $(1 \leqslant i \leqslant M)$, then $g^*(p_1, \ldots, p_{n+1}) = \min(\&(j_1(i), g_1(p_2, \ldots, p_{n+1})), \ldots, \&(j_i(i), g_i(p_2, \ldots, p_{n+1})), \ldots, \&(j_M(i), g_M(p_2, \ldots, p_{n+1}))) = \min(\&(1, g_i(p_2, \ldots, p_{n+1})) = g_i(p_2, \ldots, p_{n+1}) = g(i, p_2, \ldots, p_{n+1})$. Hence, we have $g^*(i, p_2, \ldots, p_{n+1}) = g(i, p_2, \ldots, p_{n+1})$ and our theorem follows.

This establishes the functional completeness of the many-valued system of logic which is now under study. That is, statement calculi based on C, N, and T are functionally complete. There are other many-valued statement calculi based on operators other than C, N, and T which are functionally complete [16]. Needless to say, the conditions H 1 to H 7 are such as to allow the construction of a great profusion of many-valued statement calculi which are not functionally complete. In general, however, attention has been restricted to those systems for which there are at least partial analogues of the familiar functions of two-valued logic.

When many-valued truth functions (operators) are properly analogous to two-valued truth functions (operators), we will say that they satisfy "standard conditions". More precisely, suppose that $\&(p, q)$, $\vee(p, q)$, $\supset(p, q)$, $\sim(p)$, and $j_k(p)$ $(1 \leqslant k \leqslant M)$ are many-valued truth functions. We shall say that each of these truth functions in turn satisfies standard conditions if it is such that it satisfies the following conditions:

[16] For example, see Post 1921 and Webb 1936. An interesting system for $M = 3$ is given in Słupecki 1946. Also, see Martin 1950.

(1) The value of $\&(p, q)$ is designated if and only if p and q are both designated.

(2) The value of $\mathbf{v}(p, q)$ is undesignated if and only if p and q are both undesignated.

(3) The value of $\supset(p, q)$ is undesignated if and only if p is designated and q is undesignated.

(4) The value of $\sim(p)$ is designated if and only if p is undesignated.

(5) The value of $j_k(p)$ is designated if and only if $p = k$.

Clearly, the many-valued truth functions $\&(p, q)$, $\mathbf{v}(p, q)$, $\supset(p, q)$, and $j_k(p)$ which we have already defined in terms of $f_1(p, q)$ and $f_2(p)$ satisfy standard conditions for all choices of M and S. Also, if we take the operator $\overline{}$ which has been defined in terms of f_1 and f_2 as $\sim$, then $\sim$ satisfies standard conditions for every choice of M and S. However, if f_1 and f_2 are taken as $\supset$ and $\sim$ respectively, then it will not be difficult for the reader to observe that f_1 satisfies standard conditions only when $M = 2$ and $S = 1$. If $S \geqslant 2$, then we would have $\supset(p, q) = \max(1,\ p - q + 1) = \max(1,\ (S + 1) - - (S) + 1) = 2$ when $p = S$ and $q = S + 1$. Since $S \geqslant 2$, this violates standard conditions. If $M > 2$ and $S = 1$, then we would have $\supset(p, q) = \max(1, (S+2)-(S+1)+1) = 2$ when $p = S + 1$ and $q = S + 2$. Hence, standard conditions are again violated, since $\supset(p, q)$ should be designated when p and q are both undesignated. By a similar check, our $\sim$ which is taken as f_2 satisfies standard conditions only when $M = 2S$.

In the next chapter we shall consider the problem of constructing axioms for many-valued statement calculi, and though reference will be made to standard conditions, it is advisable to note at this point that our definition of standard conditions is entirely independent of our plan for the axiomatization of many-valued statement calculi.

III

AXIOMATIZATION OF MANY-VALUED STATEMENT CALCULI

In the preceding chapter, a method was given for asserting statements of a many-valued statement calculus. Essentially the procedure was based on the use of truth-value functions which are associated with a given statement. Statements are to be asserted when and only when one of their corresponding truth-functions takes designated truth values exclusively. Thus, one may speak of "a truth-value stipulation of acceptable (assertable) statements". Such a stipulation has been made when one agrees on a choice for M, S, a set of statements, and a set of basic functions $F_i(P_1, \ldots, P_{a_i})$, as well as their corresponding truth functions $f_i(p_1, \ldots, p_{a_i})$, so that the hypotheses H 1 to H 7 are satisfied. Finally, one must agree to accept a statement when it is possible to consider it as a statement formula which is such that its corresponding truth function takes only designated values.

A truth-value stipulation has the merit of simplicity and ease of application. It also puts strongly in evidence the fact that we are dealing with a many-valued logic in which a choice has been made for M and S. However, there is an alternative procedure for accepting statements which makes no reference whatever to truth values. We are referring to the axiomatic method as it is used in modern mathematics.

The essence of the axiomatic method consists in choosing certain statements ("axioms") as acceptable along with certain paradigms ("rules") for generating other acceptable statements from given acceptable statements. Hence, by starting from the axioms and successively applying the rules, one may generate a class of acceptable statements. In axiomatic treatments of the ordinary two-valued logic, it has been customary to use only axioms and rules which are constructively specified and we shall do likewise in our axiomatic treatment of many-valued logic.

As has been indicated, we are using the term "axiom" in the

modern mathematical sense of a convenient starting point, rather than in the classical sense of a self-evident truth. Just what would constitute a self-evident truth in a many-valued logic is beyond our comprehension. Clearly, then, the choice of axioms and rules is more or less optional, if one does not have a preconceived idea of what is to constitute the class of acceptable statements. In any case, however, if we have specified a set of axioms and rules, then one may say that we have "an axiomatic stipulation of acceptable statements".

Various stipulations, either truth-value or axiomatic, generate corresponding classes of acceptable statements. Hence, it is possible to compare various stipulations by comparing their classes of acceptable statements. In particular, we shall say that a first stipulation is "as strong as" a second, if every statement which is acceptable by the second stipulation is also acceptable by the first. Further, if there is a statement which is acceptable by the first stipulation but not by the second, then we will say that the first "is stronger than" the second stipulation.

If a given axiomatic stipulation is as strong as a certain truth-value stipulation, then we will say that the given axiomatic stipulation is "deductively complete" with respect to the given truth-value stipulation. On the other hand, if the truth-value stipulation is as strong as the axiomatic stipulation, then we will say that the axiomatic stipulation is "plausible" with respect to the given truth-value stipulation. If an axiomatic stipulation is both deductively complete and plausible with respect to a truth-value stipulation, then the two stipulations are "equivalent" in the sense that both stipulations define the same set of acceptable statements.

Let $F(P)$ be some statement function built up out of the basic functions $F_i(P_1, \ldots, P_{a_i})$. Relative to a given stipulation, a statement P is said to be "F-decidable" if either P or $F(P)$ is acceptable. Also, a stipulation will be said to be "F-consistent" if there is no statement P such that both P and $F(P)$ are acceptable. If a stipulation is F-consistent, then one can say that P is rejectable (or "F-rejectable") if either $F(P)$ is acceptable or there is a Q which is acceptable such that P is $F(Q)$. Finally, a stipulation is said to be "consistent" if there is a statement which is not acceptable

according to the given stipulation. All of these ideas are generalizations of notions which are familiar in the ordinary two-valued logic.

In the two-valued logic, there are many different truth-value stipulations. For example, as basic operators one may take $\sim$ and $\vee$, $\sim$ and &, $\sim$ and $\supset$, or various forms of the stroke function. However, since M is always 2 and S is always 1, there is a unique truth-value stipulation for each set of operators (or functions) which is selected as basic, and these various truth-value stipulations are not essentially different, despite the difference of basic operators. Because of this effective uniqueness of truth-value stipulations in two-valued logic, the characteristics of axiomatic stipulations for two-valued logic are discussed as though they were absolute, rather than relative to some truth-value stipulation. Clearly, from the point of view of many-valued logic it is essential to take account of the relativity of axiomatic stipulations to truth-value stipulations in working with such notions as completeness and consistency. Note, also, that what we have called plausibility is often called consistency by writers of two-valued logic. Further, the only operator relative to which questions of decidability and consistency have been considered in two-valued logic is the operator $\sim$. The idea of general F-decidability and F-consistency has been defined [17] but not used extensively in the study of two-valued logic. Our notion of consistency has been defined for many-valued logics as have still other concepts of completeness and consistency. [18] However, we have found use for only those generalized concepts which have just been mentioned.

Since the set of statements of a truth-value stipulation contains a least statement P which is such that its only corresponding truth function is the identity function p, then a truth-value stipulation is always consistent. This follows since the value of p can be undesignated, and hence, P is not acceptable. As a corollary, each axiomatic stipulation which is plausible with respect to a truth-value stipulation is also consistent. This results since an axiomatic stipulation which is plausible relative to a consistent truth-value stipulation is also consistent.

[17] Church 1944.

[18] Post 1921.

If we have a given stipulation, then there are various ways to obtain another stipulation which is at least as strong and has a fair likelihood of being stronger than the given stipulation. To illustrate this, consider first the case of truth-value stipulations. If we have $S \leqslant M - 2$, then by H 6 it is possible to increase S. Statements which were acceptable before the increase of S remain acceptable after the increase of S, and usually new statements become acceptable after the increase of S. Also, one may obtain new acceptable statements by adding to the list of basic functions. For example, a statement calculus based on the F_1 and F_2 (C and N) of the last chapter is not functionally complete. By adding F_3 (or T) to the list of basic operators, the calculus can be made functionally complete. Hence, a truth-value stipulation based on F_1, F_2, and F_3 is stronger than one based on F_1 and F_2 alone. In this connection, one should note that a statement calculus based on only F_1 and F_2 as basic functions might still contain statements of the form $F_3(P)$. However, in such a case, only the identity function p could be asigned to $F_3(P)$, which would accordingly not be acceptable. But if $F_3(P)$ is a basic function to which the constant 2 is always assigned, then $F_3(P)$ will be acceptable in all stipulations for which $S > 1$.

Even when one already has functional completeness, it is sometimes possible to strengthen a truth-value stipulation by changing the basic functions. This may be seen by considering a case from two-valued logic. Suppose our basic statement function is P/Q with the corresponding truth function $/(p, q)$ which satisfies the following truth-value conditions:

$$/(p, q) = \begin{cases} 2 \text{ if } p \text{ and } q \text{ are both } 1. \\ 1 \text{ if } p \text{ or } q \text{ is } 2. \end{cases}$$

Let $M = 2$ and $S = 1$, and suppose that the hypotheses H 1 to H 7 are satisfied with P/Q as the sole basic function. If we define $\sim P$ as P/P and $P \,\&\, Q$ as $\sim (P/Q)$, then $\sim P$ and $P \,\&\, Q$ are statement functions with the familiar corresponding truth functions. Now, if we temporarily ignore P/Q, and count $\sim P$ and $P \,\&\, Q$ as our basic functions, then H 1 to H 7 are satisfied and we get a truth-value stipulation which is merely the familiar two-valued statement calculus based on $\sim$ and &. By the definition of $\sim$ and &, it is

clear that a statement of the form $P/\sim P$ can not be taken as a statement formula which is either of the form $\sim R_1$ or R_1 & R_2. Hence, only the identity function p can be assigned to $P/\sim P$, and since p can be undesignated, it follows that $P/\sim P$ is not acceptable in the present system of logic.

Let us now restore the function P/Q. This can not be done by simply adjoining it and at the same time retaining our functions $\sim$ and & which are defined in terms of /, since this would violate H 4. However, we can dispense with $\sim$ and & as basic functions, and take P/Q as our only basic function. Now, if in each statement formula we replace $\sim$ and & by their respective definitions in terms of /, then each statement has among its corresponding truth functions every truth function which it had before $\sim$ and & were replaced by their definitions. Hence, every statement which was previously acceptable is still acceptable. Moreover, a statement of the form $P/\sim P$ is now acceptable since it always takes a designated truth value. However, $P/\sim P$ was not previously acceptable in spite of the fact that we were dealing, as we still are, with a functionally complete statement calculus.

This sort of situation can arise when one is comparing two formulations of the two-valued statement calculus, one in terms of stroke and one in terms of $\sim$ and & (or, more commonly, one in terms of $\sim$ and $\vee$). The comparison is made by defining $\sim$ and & in terms of the stroke, and then carrying out a development in which $\sim$ and & are considered to be basic in spite of the fact that certain possible formulas (such as $P/\sim P$) cannot be expressed in terms of $\sim$ and &. Thus, in making such a comparison, special attention must be paid to formulas like $P/\sim P$ whose status depends upon whether one is taking stroke as basic or $\sim$ and & as basic. For example, this is essentially the point of Bernstein's "Remark on Nicod's Reduction of Principia Mathematica" (See Bernstein 1937).

It is usually agreed that the familiar two-valued statement calculus which is based on the operators $\sim$ and & is equivalent to the familiar two-valued statement calculus which is based on a stroke function such as /. From the previous analysis, it seems clear that difficulties will attend an attempt to give a precise definition of this form of equivalence. We will leave to others the

problem of giving a precise phrasing of such a definition for two-valued logic, as well as its generalization for many-valued statement calculi. [19]

In a later chapter we will give an axiomatic stipulation for many-valued logics which contain quantifiers, and hence, functions which are more general than the statement functions which are now under study. As in the two-valued case, this means that there are axiomatic stipulations for which there are no equivalent truth-value stipulations. [20] However, from the point of view of statement calculi, it is of even greater interest that there are axiomatic stipulations in which the forms of the axioms and rules depend entirely on statement functions and for which there is no equivalent truth-value stipulation. For example, this is the case with the system suggested by McKinsey and Tarski in which the only basic operator is the ordinary two-valued $\supset$, and the only axiom scheme is of the form $P \supset P$. [21] Such is the case also for some axiomatic stipulations of strict implication [22] and intuitionistic statement calculi. [23]

On the other hand, for each truth-value stipulation there is at least one equivalent axiomatic stipulation. This follows since one could take the set of rules as null and the set of axioms as all statements which are acceptable according to the truth-value stipulation. Such an axiomatic stipulation is trivial in the sense that the set of axioms is as large as the set of acceptable statements. However, in the present chapter we will show that there is also a non-trivial equivalent axiomatic stipulation for each of a large class of truth-value stipulations. For this purpose one could use a finite set of axioms with a rule of substitution, but we prefer to allow an infinite set of axioms which are described by means of a finite number of axiom schemes. [24] For example, letting $\supset$ be a

[19] Discussions of other matters related to the relative strengths of statement calculi defined by truth-value stipulations are to be found in such works as Wajsberg 1935, Tarski 1938, and the two articles Kalicki 1950.

[20] Church 1936.

[21] See McKinsey and Tarski 1948.

[22] See Dugundji 1940.

[23] See Gödel 1932 and McKinsey and Tarski 1948.

[24] Von Neumann 1927.

certain statement operator, we shall say that whenever P and Q are statements, then the statement $Q \supset (P \supset Q)$ is an axiom. Since there are infinitely many statements, we have an infinite set of axioms which are described by the single axiom scheme $Q \supset (P \supset Q)$.

Let us have given, then, a certain truth-value stipulation. That is, a set of statements, a set of $F_i(P_1, \ldots, P_{a_i})$, a set of $f_i(p_1, \ldots, p_{a_i})$ and values for M and S such that H 1 to H 7 are satisfied. We wish to discuss the relationship of this truth-value stipulation to a certain axiomatic stipulation (or any equivalent one) which will be given below. In particular, the rest of the present chapter will be devoted chiefly to the task of proving that there is a non-trivial axiomatic stipulation which is equivalent to our given truth-value stipulation.

To this end, choose some statement functions $P \supset Q$ and $J_k(P)$ $(1 \leqslant k \leqslant M)$ which are definable in terms of our given $F_i(P_1, \ldots, P_{a_i})$. These could be the $P \supset Q$ and $J_k(P)$ of the previous chapter which were defined in terms of $F_1(P_1, P_2)$ and $F_2(P_1)$. However, they could also be quite different statement functions which are still defined in terms of F_1 and F_2, or they could be defined in terms of a set of $F_i(P_1, \ldots, P_{a_i})$ which is different from the set F_1 and F_2. Although $P \supset Q$ and $J_k(P)$ may now have truth functions quite different from any proposed in the previous chapter, we will continue to denote these truth functions by $\supset(p_1, p_2)$ and $j_k(p_1)$ respectively.

In working with our axiomatic stipulation, use will be made of a "chain symbol" denoted by Γ, which is defined by the following set of recursive rules:

(α) If $v < u$, then $\Gamma_{i=u}^{v} P_i Q =_{\mathrm{df}} Q$.

(β) If $v \geqslant u$, then $\Gamma_{i=u}^{v} P_i Q =_{\mathrm{df}} P_v \supset \Gamma_{i=u}^{v-1} P_i Q$.

When no confusion results, we will write $\Gamma_u^v P_i Q$ for $\Gamma_{i=u}^{v} P_i Q$. Hence, $\Gamma_3^5 P_i Q$ denotes $P_5 \supset (P_4 \supset (P_3 \supset Q))$.

Let our set of axioms be generated by the following set of axiom schemes, where P, Q, R, and $P_1, \ldots, P_\beta$ are statements:

A 1. $Q \supset (P \supset Q)$.

A 2. $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$.

A 3. $(P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$.

A 4. $(J_k(P) \supset (J_k(P) \supset Q)) \supset (J_k(P) \supset Q)$.

A 5. $\Gamma_1^M (J_i(P) \supset Q) Q$.

A 6. $J_i(P) \supset P$ where $i = 1, \ldots, S$.
A 7. $\Gamma_{k=1}^{\beta} J_{p_k}(P_k) J_f(F_i(P_1, \ldots, P_\beta))$ where $i = 1, \ldots, b$, $\beta = a_i$, and $f = f_i(p_1, \ldots, p_\beta)$.

Our rule will be the following form of modus ponens:

R 1. If P and $P \supset Q$ are acceptable, then Q is acceptable, where P and Q are statements.

It should be observed that A 4 is really a set of M axiom schemes, A 6 is a set of S axiom schemes, and A 7 is a set of M^β axiom schemes for each i. Thus, this particular axiomatic stipulation depends explicitly on the choice of S and M. There are other axiomatic stipulations in which no explicit indication is made for S and M, and some for which no particular choice of S and M seems appropriate since they are not equivalent to any truth-value stipulation.

In what follows it will be convenient to introduce a "yields sign" $\vdash$ which we will now define:

If $P_1, \ldots, P_n$, and Q are statements, then "$P_1, \ldots, P_n \vdash Q$" shall mean that there is a sequence of steps $\mathbf{s}_1, \ldots, \mathbf{s}_q$ which satisfies the following conditions:

(1) $\mathbf{s}_q$ is Q

(2) $\mathbf{s}_i$ $(1 \leqslant i \leqslant q)$ is either P_j $(1 \leqslant j \leqslant n)$ or $\mathbf{s}_i$ is an axiom or $\mathbf{s}_i$ is an $\mathbf{s}_r$ $(r < i)$ or $\mathbf{s}_i$ is R where $\mathbf{s}_s$ $(s < i)$ is $\mathbf{s}_t \supset R$ $(t < i$ and $t \neq s)$. In this last case we shall say that rule R 1 is applied to $\mathbf{s}_t$ and $\mathbf{s}_s$ to obtain R. Also, we will refer to a sequence of steps $\mathbf{s}_1, \ldots, \mathbf{s}_q$ which satisfies conditions (1) and (2) of our definition of a yields sign as "a demonstation of $P_1, \ldots, P_n \vdash Q$".

In the present definition, "an axiom" is defined by the axiom schemes A 1 to A 7. Later we shall alter our axiom schemes and introduce rules other than R 1. However, in such cases of a change of axioms and rules, it will not be difficult for the reader to make the necessary changes in our present definition of a yields sign. Hence, in what follows it will be assumed that our definition of a yields sign is altered to take account of any change that is made in our choice of axioms and rules of inference. That is, we will not give an explicit definition of a yields sign after each change of axioms and rules, but the reader should take the definition as implicit in such cases. It will be useful to note also that "$P_1, \ldots, P_n \vdash Q$" is in effect the same as "Q is acceptable under the assump-

tions $P_1, \ldots, P_n$", and, hence, "$\vdash Q$" is the same as "Q is acceptable". In each case, of course, acceptability will be relative to a choice of axioms and rules. When we prove that "Q is acceptable" relative to an axiomatic stipulation, either the sequence of steps $\mathfrak{s}_1, \ldots, \mathfrak{s}_q$ will be constructed or explicit directions will be given for constructing such a sequence. Hence, to this extent our proofs of the acceptability of a statement Q relative to an axiomatic stipulation will be constructive.

We are now in a position to prove some theorems, the first of which will establish the deductive completeness of our axiomatic stipulation. To this end, some lemmas will first be proved.

Lemma 3. 1. 1. $\vdash (Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$.

Proof. In A 2 take $P \supset Q$, $Q \supset R$, and $P \supset R$ as instances of P, Q, and R respectively. This gives,

(1) $\vdash ((P \supset Q) \supset ((Q \supset R) \supset (P \supset R))) \supset ((Q \supset R) \supset ((P \supset Q) \supset (P \supset R)))$.

If we now use (1) with A 3 our lemma follows by R 1.

Lemma 3. 1. 2. $\vdash Q \supset Q$.

Proof. In A 2 take Q, P, and Q as instances of P, Q, and R respectively. This gives,

(1) $\vdash (Q \supset (P \supset Q)) \supset (P \supset (Q \supset Q))$.

Hence, A 1 with (1) and R 1 gives (2), $\vdash P \supset (Q \supset Q)$. If we now take A 1 as an instance of P, our lemma follows by R 1 and (2).

Lemma 3. 1. 3. $\vdash (P \supset Q) \supset ((\Gamma_1^n S_i P) \supset \Gamma_1^n S_i Q)$.

Proof. Use induction on n.

(α) If $n = 0$, use lemma 3. 1. 2.

(β) Assume our theorem for $n = k$ and prove it for $n = k + 1$. By lemma 3. 1. 1 and the definition of a chain symbol we have,

(1) $\vdash ((\Gamma_1^k S_i P) \supset \Gamma_1^k S_i Q) \supset ((\Gamma_1^{k+1} S_i P) \supset \Gamma_1^{k+1} S_i Q)$.

Now use assumption (β) and A 3.

Lemma 3. 1. 4. Let $Q_1, \ldots, Q_q$ denote an ordered set of statements among which each of $P_1, \ldots, P_p$ occurs at least once. Then, $\vdash (\Gamma_1^p P_i R) \supset \Gamma_1^q Q_i R$.

Proof. Use induction on q.

(α) If $q = 0$ then $p = 0$, so use lemma 3. 1. 2.

(β) Assume the theorem for $q = k$ and prove it for $q = k + 1$.

Case 1. If $p = 0$, then by assumption (β) we have,

(1) $\vdash R \supset \Gamma_1^k Q_i R$.

Using A 1 and the definition of a chain symbol we can get,

(2) $\vdash \Gamma_1^k Q_i R \supset \Gamma_1^{k+1} Q_i R$.

Now use (1), (2), and A 3.

Case 2. If $p > 0$, then P_p occurs among $Q_1, \ldots, Q_{k+1}$. Suppose P_p is Q_j $(1 \leqslant j \leqslant k+1)$. By assumption ($\beta$) we then have,

(3) $\vdash (\Gamma_1^{p-1} P_i R) \supset \Gamma_{j+1}^{k+1} Q_i (\Gamma_1^{j-1} Q_i R)$.

But by lemma 3. 1. 1 and the definition of a chain symbol we have,

(4) $\vdash (\Gamma_1^p P_i R) \supset (Q_j \supset \Gamma_{j+1}^{k+1} Q_i (\Gamma_1^{j-1} Q_i R))$.

Subcase 1. If $j = k+1$, then our theorem follows by (4) and the definition of a chain symbol.

Subcase 2. If $j < k+1$, then by A 2 we have,

(5) $\vdash (Q_j \supset \Gamma_{j+1}^{k+1} Q_i (\Gamma_1^{j-1} Q_i R)) \supset (Q_{k+1} \supset (Q_j \supset \Gamma_{j+1}^{k} Q_i (\Gamma_1^{j-1} Q_i R)))$.

But by assumption (β) we get,

(6) $\vdash (Q_j \supset \Gamma_{j+1}^{k} Q_i (\Gamma_1^{j-1} Q_i R)) \supset \Gamma_1^k Q_i R$.

Hence, by (6), lemma 3. 1. 1, and the definition of a chain symbol we have,

(7) $\vdash (Q_{k+1} \supset (Q_j \supset \Gamma_{j+1}^{k} Q_i (\Gamma_1^{j-1} Q_i R))) \supset \Gamma_1^{k+1} Q_i R$.

If we now use (4), (5), and (7) with A 3 our lemma follows.

Lemma 3. 1. 5. $\vdash (J_k(P) \supset (Q \supset R)) \supset ((J_k(P) \supset Q) \supset (J_k(P) \supset R)))$.

Proof. By A 2 we have,

(1) $\vdash (J_k(P) \supset (Q \supset R)) \supset (Q \supset (J_k(P) \supset R))$.

Also, lemma 3. 1. 1 gives,

(2) $\vdash (Q \supset (J_k(P) \supset R)) \supset ((J_k(P) \supset Q) \supset (J_k(P) \supset (J_k(P) \supset R)))$.

But by A 4 we have,

(3) $\vdash (J_k(P) \supset (J_k(P) \supset R)) \supset (J_k(P) \supset R)$.

Hence, using (3) with lemma 3. 1. 1 we can get,

(4) $\vdash ((J_k(P) \supset Q) \supset (J_k(P) \supset (J_k(P) \supset R))) \supset ((J_k(P) \supset Q) \supset$
$\supset (J_k(P) \supset R))$.

If we now use (1), (2), and (4) with A 3 our theorem follows.

Lemma 3. 1. 6. $\vdash (\Gamma_{r=1}^p J_{e_r}(P_r) (R \supset S)) \supset ((\Gamma_{r=1}^p J_{e_r}(P_r) R) \supset$
$\supset \Gamma_{r=1}^p J_{e_r}(P_r) S)$.

Proof. Use induction on p.

(α) If $p = 0$, then use lemma 3. 1. 2.

(β) Assume the theorem for $p = k$ and prove it for $p = k+1$.

By assumption (β) and lemma 3. 1. 1 we can get,

(1) $\vdash (J_{e_{k+1}}(P_{k+1}) \supset (\Gamma_{r=1}^{k} J_{e_r}(P_r)(R \supset S))) \supset (J_{e_{k+1}}(P_{k+1}) \supset$
$\supset ((\Gamma_{r=1}^{k} J_{e_r}(P_r)R) \supset \Gamma_{r=1}^{k} J_{e_r}(P_r)S))$.

Also, lemma 3. 1. 5 and the definition of a chain symbol will give,

(2) $\vdash (J_{e_{k+1}}(P_{k+1}) \supset ((\Gamma_{r=1}^{k} J_{e_r}(P_r)R) \supset \Gamma_{r=1}^{k} J_{e_r}(P_r)S)) \supset$
$\supset ((\Gamma_{r=1}^{k+1} J_{e_r}(P_r)R) \supset \Gamma_{r=1}^{k+1} J_{e_r}(P_r)S)$.

But by the definition of a chain symbol and (1) we have,

(3) $\vdash (\Gamma_{r=1}^{k+1} J_{e_r}(P_r)(R \supset S)) \supset (J_{e_{k+1}}(P) \supset ((\Gamma_{r=1}^{k} J_{e_r}(P_r)R) \supset$
$\supset \Gamma_{r=1}^{k} J_{e_r}(P_r)S))$.

If we now use (3), (2), and A 3 our lemma follows.

Lemma 3. 1. 7. Suppose that W is a statement formula which is built up out of $Q_1, \ldots, Q_p$. Let $w(q_1, \ldots, q_p)$ be the corresponding truth-value function of W. Then, if w denotes $w(q_1, \ldots, q_p)$ we have,

$$\vdash \Gamma_{t=1}^{p} J_{q_t}(Q_t) J_w(W).$$

Proof. Use induction on the number of symbols in W. Let W have k symbols and assume the theorem for any W with less than k symbols.

Case 1. If W is a Q_t ($1 \leqslant t \leqslant p$), then w is q_t and by lemma 3. 1. 4 we have,

(1) $\vdash (J_w(W) \supset J_w(W)) \supset \Gamma_{t=1}^{p} J_{q_t}(Q_t) J_w(W)$.

Now use lemma 3. 1. 2.

Case 2. Suppose that W is $F_i(V_1, \ldots, V_{a_i})$ and let v_j denote the truth function corresponding to V_j ($1 \leqslant j \leqslant a_i$). Then, $w = f_i(v_1, \ldots, v_{a_i})$. By H 3 and assumption (β) we have,

(2) $\vdash \Gamma_{t=1}^{p} J_{q_t}(Q_t) J_{v_j}(V_j)$.

But by A 7 we can get,

(3) $\vdash \Gamma_{j=1}^{\beta} J_{v_j}(V_j) J_w(W)$ where β is a_i.

Using (3), lemma 3. 1. 3, and the definition of a chain symbol we have,

(4) $\vdash \Gamma_{t=1}^{p} J_{q_t}(Q_t) J_{v_\beta}(V_\beta) \supset \Gamma_{t=1}^{p} J_{q_t}(Q_t) (\Gamma_{j=1}^{\beta-1} J_{v_j}(V_j) J_w(W))$.

Hence, by (2) and the definition of a chain symbol we can get,

(5) $\vdash \Gamma_{t=1}^{p} J_{q_t}(Q_t) (J_{v_{\beta-1}}(V_{\beta-1}) \supset \Gamma_{j=1}^{\beta-2} J_{v_j}(V_j) J_w(W))$.

Now, using (5) with lemma 3. 1. 6, and then using (2) with the definition of a chain symbol we can get,

(6) $\vdash \Gamma_{t=1}^{p} J_{q_t}(Q_t) (J_{v_{\beta-2}}(V_{\beta-2}) \supset \Gamma_{j=1}^{\beta-3} J_{v_j}(V_j) J_w(W))$.

Clearly, by continuing the argument used to obtain (6) from (5) we could finally prove our theorem.

Lemma 3. 1. 8. If W is as in lemma 3. 1. 7 and the value of w is always designated, then $\vdash W$.

Proof. By A 6 and lemma 3. 1. 3 we have,

(1) $\vdash \Gamma^{p}_{t=1} J_{q_t}(Q_t)\, J_w(W) \supset \Gamma^{p}_{t=1} J_{q_t}(Q_t)\, W$.

Hence, lemma 3. 1. 7 and the definition of a chain symbol with (1) will give the following for each set of q_p's,

(2) $\vdash J_{q_p}(Q_p) \supset \Gamma^{p-1}_{t=1} J_{q_t}(Q_t)\, W$ where $q_p = 1, \ldots, M$.

Now using (2) with A 5 one can get,

(3) $\vdash \Gamma^{p-1}_{t=1} J_{q_t}(Q_t)\, W$.

Hence, by (3) and the definition of a chain symbol we have for each set of q_{p-1}'s,

(4) $\vdash J_{q_{p-1}}(Q_{p-1}) \supset \Gamma^{p-2}_{t=1} J_{q_t}(Q_t)\, W$ where $q_{p-1} = 1, \ldots, M$.

Clearly, by continuing the argument used to obtain (4) from (2) we could finally get $\vdash W$.

Theorem 3. 1. The axiomatic stipulation given by A 1 to A 7 and R 1 is deductively complete relative to our given truth-value stipulation.

Proof. Let Y be any statement which is acceptable according to our given truth-value stipulation. Then Y may be considered as a statement formula W which is built up out of $Q_1, \ldots, Q_p$ in such a manner that the corresponding truth function w of W always takes a designated truth-value. Hence, by lemma 3. 1. 8 we have $\vdash W$, which means that Y is acceptable according to our axiomatic stipulation. It follows that our axiomatic stipulation is as strong as our truth-value stipulation, and thus we have our theorem.

Next we will wish to consider the problem of plausibility for our axiomatic stipulation, but this will require some additional definitions and lemmas.

Definition of "a truth function corresponding to an axiom scheme": Each axiom scheme, as explicitly stated, may be taken as a statement formula which is built up out of P, Q, etc. which appear in the written form of the axiom scheme. The truth function corresponding to this statement formula is the truth function

corresponding to the axiom scheme. For example, the truth function corresponding to A 1 is $\supset (q, \supset (p, q))$.

Definition of "the set of primary constituents of a statement":

Case 1. If W is a statement which is not of the form $F_i(P_1, \ldots, P_{a_i})$ for any i or set of $P_1, \ldots, P_{a_i}$, then W is the one and only primary constituent of W.

Case 2. If W is a statement of the form $F_i(P_1, \ldots, P_{a_i})$ for some i and some set of $P_1, \ldots, P_{a_i}$ then let $R_{j1}, \ldots. R_{j\beta_j}$ denote the primary consituents of $P_j (1 \leqslant j \leqslant a_i)$. The totality of all such $R_{j1}, \ldots, R_{j\beta_j}$ after removing duplications, is the set of primary constituents of W.

Lemma 3. 2. 1. Each statement is a unique statement formula of its primary constituents.

Proof. Let W denote any statement and use induction on the structure of W.

Case 1. If W is not of the form $F_i(P_1, \ldots, P_{a_i})$, then our theorem follows by case 1 of the definition of a set of primary constituents of a statement.

Case 2. If W is of the form $F_i(P_1, \ldots, P_{a_i})$, then our assumption of induction would assure us that each P_j $(1 \leqslant j \leqslant a_i)$ is a unique statement formula of its primary consitutents. But by H 4, W is a unique statement formula of $P_1, \ldots, P_{a_i}$. Hence, our theorem follows by case 2 of the definition of a primary constituent of a statement.

In view of lemma 3. 2. 1, one can consider a statement as a statement formula of its primary constituents. The truth function which corresponds to this statement formula will be called "the primary truth function of the given statement". Note that by definition, if f and g are the primary truth functions for P and Q respectively, then $\supset (f, g)$ is the primary truth function for $P \supset Q$.

Lemma 3. 2. 2. If W is a statement which is an instance of an axiom scheme and the truth function corresponding to the axiom scheme takes only designated values, then the primary truth function of W takes designated values exclusively.

Proof. If W is an instance of A 1, then by H 4 there are statements S_1 and S_2 such that W is the same statement as $S_1 \supset (S_2 \supset S_1)$. Let w be the primary truth function of W, and suppose that s_1 and s_2 are the primary truth functions of S_1 and S_2. Then we see that

$w = \supset (s_1, \supset (s_2, s_1))$. Now, whatever values are assigned to the variables in s_1 and s_2, we get a pair of values for s_1 and s_2. Hence, the value of $\supset (s_1, \supset (s_2, s_1))$ is just the value that one gets by letting $q = s_1$ and $p = s_2$ in the truth function $\supset (q, \supset (p, q))$ corresponding to A 1. But we are assuming the truth function corresponding to each axiom scheme takes only designated values. Hence, the value of $\supset (s_1, \supset (s_2, s_1))$ is designated for each assignment of values to the variables in s_1 and s_2.

Clearly, by a similar kind of argument, our lemma will follow for any case where W is an instance of an axiom scheme other than A 1.

Definition of "plausible truth functions": The truth functions $\supset (p, q)$ and $j_k(p)$ are said to be plausible [25] if they satisfy the following two conditions:

(1) They are such that the truth function corresponding to each axiom scheme A 1 to A 7 takes designated truth values exclusively.

(2) If p is designated and q is undesignated, then $\supset (p, q)$ is undesignated.

Lemma 3. 2. 3. If $\supset (p, q)$ and $j_k(p)$ are plausible and $\vdash W$, then the primary truth function of W takes designed truth values exclusively.

Proof. Use induction on the number of steps in the demonstration of $\vdash W$.

(α) By the definition of a demonstration, if the demonstration of $\vdash W$ consists of a single step, then the step must be an axiom which is the same as W. Hence, W is an instance of some one of our axiom schemes A 1 to A 7, and by (1) of the definition of plausible truth functions with lemma 3. 2. 2 we know that the primary truth function of W takes designated values exclusively and our lemma follows for case (α).

(β) Assume our lemma for all demonstrations of $\vdash W$ with less than k steps and prove it for demonstrations of $\vdash W$ with k steps. Let $\mathbf{s}_1, \ldots, \mathbf{s}_k$ be a demonstration of $\vdash W$ with k steps. By the definition of a demonstration, W is $\mathbf{s}_k$ and we have three cases to consider.

[25] Note the difference between plausible functions and plausible stipulations.

Case 1. If $\mathbf{s}_k$ is an axiom, our lemma follows by the same argument that was used in (α).

Case 2. If $\mathbf{s}_k$ is the same as an earlier step $\mathbf{s}_j$ ($j < k$), then $\mathbf{s}_1, \ldots, \mathbf{s}_j$ is a demonstration of $\vdash W$ with less than k steps. Hence, our lemma follows by assumption (β).

Case 3. If R 1 is applied to two earlier steps $\mathbf{s}_t \supset \mathbf{s}_k$ and $\mathbf{s}_t$ to obtain $\mathbf{s}_k$, then there are demonstrations of $\vdash \mathbf{s}_t \supset \mathbf{s}_k$ and of $\vdash \mathbf{s}_t$ which have less than k steps. Hence, by (β) the primary truth function of $\mathbf{s}_t \supset \mathbf{s}_k$ and of $\mathbf{s}_t$ takes designated truth values exclusively. Let $\supset (s_t, s_k)$, s_t, and s_k denote the primary truth functions of $\mathbf{s}_t \supset \mathbf{s}_k$, $\mathbf{s}_t$, and $\mathbf{s}_k$ respectively. If s_k does not take designated truth values exclusively, then by (2) of the denfinition of plausible truth functions, we know that $\supset (s_t, s_k)$ does not always take designated values. But this contradicts our previous result that $\supset (s_t, s_k)$ takes designated truth values exclusively. Hence, s_k must always take designated values and our lemma follows.

Theorem 3. 2. If $\supset (p, q)$ and $j_k(p)$ are plausible, then our axiomatic stipulation is plausible relative to our truth value stipulation.

Proof. By lemma 3. 2. 3, if W is acceptable according to our axiomatic stipulation, then it is acceptable according to our truth-value stipulation. Hence, the latter stipulation is as strong as the former and our theorem follows.

Theorem 3. 3. If $\supset (p, q)$ and $j_k(p)$ are plausible, then our axiomatic stipulation is equivalent to our truth-value stipulation in the sense that the class of acceptable statements defined by each stipulation is the same.

Proof. This is an immediate consequence of theorems 3. 1 and 3. 2.

Theorem 3. 4. If $\supset (p, q)$ and $j_k(p)$ satisfy standard conditions, then our axiomatic stipulation is equivalent to our truth-value stipulation.

Proof. By checking A 1 through A 7 and using the definition of standard conditions for the functions $\supset (p, q)$ and $j_k(p)$, it is not difficult to show that conditions (1) and (2) in the definition of plausible truth functions are satisfied by the truth functions $\supset (p, q)$ and $j_k(p)$ which satisfy standard conditions. Hence, our theorem will follow by theorem 3. 3.

As a result of theorem 3. 4, we can say that if $\supset (p, q)$, $\sim (p)$, and $j_k(p)$ satisfy standard conditions, then our stipulations are $\sim$-consistent. Also, if P is a statement which is such that its primary truth function is either always designated or always undesignated, then P is $\sim$-decidable. That is, if P is such that we have either $\vdash P$ or $\vdash \sim P$, then P is $\sim$-decidable.

Theorem 3. 5. If in A 1 to A 7 we use the $F_1(P, Q)$ of the previous chapter for $P \supset Q$ and use the $J_k(P)$ which was defined in terms of F_1 and F_2 in the previous chapter, then our axiomatic stipulation is equivalent to our truth-value stipulation when $S = 1$.

Proof. Recalling that $f_1(p, q) = \max(1, q - p + 1)$ and that $j_k(p) = 1$ if $p = k$ and $j_k(p) = M$ if $p \neq k$, we can check each of A 1 to A 7 and show that condition (1) of the definition of the plausibility of $\supset (p, q)$ and $j_k(p)$ is satisfied. Since $S = 1$, if $p \leqslant S$ and $q > S$, then $\supset (p, q) = f_1(p, q) = \max(1, q) = q > S$. Hence, condition (2) of the definition of the plausibility of $\supset (p, q)$ and $j_k(p)$ is satisfied. Our theorem thus follows by theorem 3. 3.

Since F_1 and F_2 are the Łukasiewicz–Tarski C and N, theorem 3. 5 indicates that we can use A 1 to A 7 and R 1 with C for $\supset$ to get an axiomatic stipulation for statement calculi based on C and N when $S = 1$. Such a stipulation has the interesting property that the truth function $C(p, q)$ which corresponds to the operator $\supset$ which appears in the axiom schemes does not satisfy standard conditions for $M > 2$. This illustrates the point that though standard conditions for the truth function which corresponds to the $\supset$ which appears in the axiom schemes are often useful they are by no means necessary.

However, although $C (p, q)$ does not satisfy standard conditions for $M > 2$, we have remarked in the proof of theorem 3. 5 that the truth functions $C (p, q)$ and $j_k(p)$ are plausible when $S = 1$. It is interesting to note that these truth functions are not plausible for $S > 1$. Since $\supset$ is now C, we have $\supset (p, q) = \max (1, q - p + 1)$. Hence, if $p = S$ and $q = S + 1$, then $\supset (p, q) = 2$. Since $S > 1$, this violates condition (2) in the definition of plausible truth functions. This indicates that if A 1 to A 7 and R 1 are to be used to axiomatize Łukasiewicz–Tarski statement calculi based on C and N, then for $S > 1$ we must choose an $\supset$ different from C. However, this presents no difficulty since we could take $P \supset Q$ to be the $\overline{P} A Q$

as indicated in the last chapter. Then $\supset(p, q)$ satisfies standard conditions, and since $J_k(P)$ can be so defined in terms of C and N that its corresponding truth function $j_k(p)$ satisfies standard conditions, we know by theorem 3. 4 that the desired axiomatization can be obtained. In fact, this same $\supset$ and $J_k(\ )$ could be used to give an axiomatization of statement calculi based on C and N even when $S = 1$. This is the import of the next theorem.

Theorem 3. 6. If our basic functions are $F_1(P_1, P_2)$ and $F_2(P_1)$, then for $S \geqslant 1$ there are $\supset$ and $J_k(\ )$ $(1 \leqslant k \leqslant M)$ which are such that our axiomatic stipulation is equivalent to our truth-value stipulation.

Proof. If we take $\supset$ and $J_k(\ )$ as indicated above, then our theorem follows by theorem 3. 4.

In the light of theorems 3. 5 and 3. 6, it is clear that our method of formalizing many-valued logics is applicable to logics which are based on a set of functionally incomplete operators. It is not difficult to see that similar results may be obtained when the basic set of operators is functionally complete.

Theorem 3. 7. If our set of basic functions $F_i(P_1, \ldots, P_{a_i})$ is functionally complete, then for $S \geqslant 1$ there are $\supset$ and $J_k(\ )$ $(1 \leqslant k \leqslant M)$ for which our axiomatic stipulation is equivalent to our truth-value stipulation.

Proof. Since our basic functions are functionally complete, we can define $P \supset Q$ and $J_k(P)$ so that their corresponding truth functions satisfy standard conditions. Hence, our theorem follows by theorem 3. 4.

It is of interest to note that if our basic functions are functionally complete, then we can so define $\sim P$ that our stipulations are $\sim$-consistent. In fact, if $\sim P$ is defined as in the proof of theorem 3. 6, then our stipulations remain $\sim$-consistent even if the functionally incomplete set of operators C and N are taken as basic. Similar remarks hold regarding the $\sim$-decidability of statements whose primary truth functions are either always designated or always undesignated. Note, however, that if $M > 2$ and $S > 1$, then if C and N are our basic functions, our stipulations may not be N-consistent, and statements whose primary truth functions are either always designated or undesignated may not be N-decidable. For example, if $M = 3$ and $S = 2$, then P and NP may both be acceptable.

One can often give a more elegant axiomatic stipulation which is equivalent to that using A 1 to A 7 and R 1 by proceeding as follows: [26]

(1) Replace A 2, A 3, and A 4 by the single axiom scheme,

B. $(P \supset Q) \supset ((P \supset (Q \supset R)) \supset (P \supset R))$.

(2) Retain A 1, A 5, A 6, A 7, and R 1. Let us refer to this new axiomatic stipulation defined by (1) and (2) as the B-stipulation in contrast to the stipulation defined by A 1 to A 7 and R 1 which will be called the A-stipulation. The following results are obtained by the B-stipulation.

Lemma 3. 8. 1. $\vdash P \supset P$.

Proof. In B, if we take P as an instance of R, and $(Q \supset P)$ as an instance of Q we have,

(1) $\vdash (P \supset (Q \supset P)) \supset ((P \supset ((Q \supset P) \supset P)) \supset (P \supset P))$.

Hence, A 1 and R 1 with (1) give,

(2) $\vdash (P \supset ((Q \supset P) \supset P)) \supset (P \supset P)$.

Using A 1 and R 1 again with (2) gives our theorem.

Theorem 3. 8 (Deduction theorem). If $P_1, \ldots, P_n \vdash Q$, then $P_1, \ldots, P_{n-1} \vdash P_n \supset Q$.

Proof. Use A 1, B, and lemma 3. 8. 1 as in the usual proof of the deduction theorem for statement calculi in the two-valued case. [27]

Lemma 3. 9. 1. $\vdash$ A 2.

Proof. By R 1 we have,

(1) $P \supset (Q \supset R),\ Q,\ P \vdash Q \supset R$.

Hence, by R 1 again we have,

(2) $P \supset (Q \supset R),\ Q,\ P \vdash R$.

Our theorem follows from (2) by three uses of the deduction theorem.

Lemma 3. 9. 2. $\vdash$ A 3.

Proof. By two uses of R 1 we get,

(1) $P \supset Q, Q \supset R,\ P \vdash R$.

Our theorem follows from (1) by three uses of the deduction theorem.

[26] Kalmár 1934.

[27] See Church 1944, pp. 45–46.

Lemma 3. 9. 3. $\vdash$ A 4.

Proof. By R 1 we have,

(1) $J_k(P) \supset (J_k(P) \supset Q),\ J_k(P) \vdash J_k(P) \supset Q.$

Hence, by R 1 again we have,

(2) $J_k(P) \supset (J_k(P) \supset Q),\ J_k(P) \vdash Q.$

Our theorem follows by two uses of the deduction theorem.

Theorem 3. 9. Our B-stipulation is as strong as our A-stipulation.

Proof. This is an immediate consequence of lemmas 3. 9. 1, 3. 9. 2, 3. 9. 3 and the definition of a B-stipulation.

Theorem 3. 10. There are definitions of $\supset$ which make our B-stipulation stronger than our A-stipulation.

Proof. In our B-stipulation, the axiom scheme B is acceptable for all $\supset$'s which are defined in terms of the functions $F_i(P_1, \ldots, P_{a_i})$. However, there are definitions of $\supset$ for which B is not acceptable in our A-stipulation. For example, take $S = 1$ and the $F_1(P_1, P_2)$ and $F_2(P_1)$ of the previous chapter as our $F_i(P_1, \ldots, P_{a_i})$. If we now define $\supset$ as F_1, then B is not always acceptable according to our truth-value stipulation. In particular, let P, Q, R take the truth values 3, 2, 4 respectively when $M = 4$ and $S = 1$. Hence, by theorem 3. 5, B is not always acceptable in our A-stipulation as it is in our B-stipulation and our theorem follows.

Corollary. There is an axiomatic stipulation which is equivalent to a truth-value stipulation, but in which the deduction theorem is not provable.

Proof. If the deduction theorem were provable in the A-stipulation, then axiom scheme B would be acceptable in the A-stipulation, since axiom scheme B is an easy consequence of the deduction theorem. However, for some definitions of $P \supset Q$, axiom scheme B is not acceptable in our A-stipulation. In particular, it is not acceptable in the example of an A-stipulation given in the proof of theorem 3. 10, which by theorem 3. 5 is equivalent to the truth-value stipulation defined by taking F_1 and F_2 as basic when $M = 4$ and $S = 1$.

With the proof of theorem 3. 10 we can now see that an A-stipulation is not always equivalent to a B-stipulation. However, the two stipulations may often be equivalent. For example, they are

equivalent when the truth functions which correspond to the $\supset$ and $J_k(\ )$ which appear in the axiom schemes of the two stipulations satisfy standard conditions. This is easily verified by referring to theorems 3. 9 and 3. 4. Also, under the same circumstances both our A-stipulation and our B-stipulation will be equivalent to our given truth-value stipulation. For example, this will be the case if $\supset$ and $J_k(\ )$ are our basic operators and their corresponding truth functions $\supset (p, q)$ and $j_k(p)$ satisfy standard conditions.

We have already remarked that there are axiomatic stipulations for certain statement calculi which are not equivalent to any given truth-value stipulation. As one of our examples of such an axiomatic stipulation, we referred to some systems of strict implication. We are now in a position to study such systems as that of strict implication from the point of view of our treatment of many-valued logics. In order to illustrate such a treatment, use will be made of the following matrix: [28]

(***m*** 1)

&	1	2	3	4	$\sim$	$\diamond$
1	1	2	3	4	4	1
2	2	2	4	4	3	1
3	3	4	3	4	2	1
4	4	4	4	4	1	4

Theorem 3. 11. If we take &, $\sim$, and $\diamond$ as our basic operators, then $P \supset Q$ and $J_k(P)$ can be so defined that their corresponding truth functions $\supset (p, q)$ and $j_k(p)$ satisfy standard conditions for $M = 4$ and $S = 2$.

Proof. Let the truth-value functions which correspond to our basic functions $P_1 \,\&\, P_2$, $\sim P_1$, and $\diamond P_1$ be defined by the matrix ***m*** 1. Now take the following definitions for $P \supset Q$ and $J_k(P)$:

$$
\begin{aligned}
P \supset Q &=_{\mathrm{df}} \sim (P \,\&\, \sim Q) \\
J_1(P) &=_{\mathrm{df}} \sim \diamond \sim P \\
J_2(P) &=_{\mathrm{df}} P \,\&\, \diamond \sim P \\
J_3(P) &=_{\mathrm{df}} J_2(\sim J_2(P)) \\
J_4(P) &=_{\mathrm{df}} \sim \diamond\, P
\end{aligned}
$$

[28] Lewis and Langford 1932, p. 493.

Since $M = 4$ and $S = 2$, a simple check using ***m*** 1 will indicate that our theorem follows.

Theorem 3. 12. If &, $\sim$, and $\Diamond$ are as in theorem 3. 11, then for $M = 4$ and $S = 2$ there is an axiomatic stipulation with &, $\sim$, and $\Diamond$ as basic operators which is equivalent to our truth-value stipulation based on ***m*** 1.

Proof. This is an immediate consequence of theorems 3. 4 and 3. 11.

In the usual treatment of strict implication [29] it is customary to define $P ⥽ Q$ as $\sim \Diamond (P \,\&\, \sim Q)$ and to use ⥽ in the role of $\supset$. We could not let ⥽ serve as the $\supset$ which appears in the axiom schemes A 1 to A 7. For if ⥽ (p_1, p_2) denotes the truth function corresponding to $P_1 ⥽ P_2$, then by ***m*** 1 we get ⥽ (2, ⥽ (1, 2)) = 4, so that the truth function corresponding to A 1 takes an undesignated value. Hence, the axiomatic stipulation based on A 1 and A 7 and R 1 is not plausible with ⥽ playing the role of $\supset$. However, a plausible axiomatic stipulation with ⥽ playing the role of $\supset$ can be given by changing our set of axiom schemes. For instance, in place of A 1 to A 7 one could use Lewis and Langford's [30] B 1 to B 8 and rewrite R 1 with ⥽ in the place of $\supset$. By reference to ***m*** 1, it can be shown that this latter axiomatic stipulation is plausible with respect to the truth-value stipulation defined by ***m*** 1 with $M = 4$ and $S = 2$. [31] But, with respect to this same truth-value stipulation, the axiomatic stipulation using B 1 to B 8 is not deductively complete. In particular, the "Brouwersche axiom" [32] has a truth function which takes only designated values, yet this axiom is not derivable [33] in the axiomatic stipulation based on B 1 to B 8. Indeed, this latter axiomatic stipulation is one of those for strict implication which has no equivalent truth-value stipulation. [34] Hence, we can say that such axiomatic stipulations are never both deductively complete and plausible. This indicates that the usual formulations or axiomatic stipulations

[29] Ibid, p. 124.
[30] Ibid, p. 493.
[31] Ibid, pp. 493—495.
[32] Ibid, p. 497.
[33] Ibid, p. 498.
[34] Dugundji 1940.

for strict implication and intuitionistic logic, since they are never deductively complete, are in sharp contrast with the systems of logic that are the chief objects of study in this work. These latter are based on stipulations equivalent to our A-stipulation and these are always deductively complete. [35]

[35] For some further connections of systems of strict implication with many-valued logic, see Halldén 1949 and Bergmann 1949.

IV

FOUNDATIONS OF THE THEORY OF QUANTIFICATION FOR MANY-VALUED LOGICS

We are now in a position to give a method for introducing quantifiers into many-valued logics, that is to develop the theory of many-valued predicate calculi. As we shall see, just as our treatment of the statement calculus for the case of many truth values is a generalization of the familiar treatment for the case of two truth values, so our treatment of the predicate calculus for the many-valued case is a generalization of the familiar treatment for the case of two truth values. [36]

In addition to statements and functions of statements which have already been introduced, we now introduce individual variables denoted by X, Y, Z, X_1, Y_1, $\ldots$, and predicates denoted by $F(X)$, $G(Y_1, \ldots, Y_n)$, $H(X, Y, Z)$, $\ldots$. Both individual variables and predicates are allowed as constituents of statement formulas, but only the predicates are counted as statements. That is, predicates can be combined with other predicates and other statements to form new statements by using the statement functions which are already given. This is not the case with individual variables. In terms of meaning, it is customary to think of the individual variables as taking values from some universe of individuals. Then, the predicates are thought of as truth functions of the individual variables in the sense that if values from the universe of individuals are assigned to each of $X_1, \ldots, X_n$, then the predicate $F(X_1, \ldots, X_n)$ is a statement with a correspondingly determined truth value.

In addition to individual variables and predicates, quantifiers are adjoined to our set of basic symbols. The two-valued prototypes of our many-valued quantifiers are the familiar "universal quantifier" $(X)F(X)$ and "existential quantifier" $(\exists X)F(X)$. However, the theory of quantification which is developed here will

[36] For example, see Church 1944 or Hilbert and Ackermann 1949.

allow a more generalized form of quantifier which permits the binding of several variables at once, and the combining of more than one statement (or predicate) in the process of quantification. With these preliminary remarks in mind, we will now add to our hypotheses H 1 to H 7.

H 8. Among the symbols permitted for use in constructing formulas, there is a denumerably infinite set of symbols which are called individual variables and denoted by X, Y, Z, etc., with or without subscripts. No individual variable is a statement, although individual variables may occur as parts of statements.

H 9. There are c basic functions denoted by $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ with $1 \leqslant i \leqslant c$, $1 \leqslant \beta_i$, and $1 \leqslant \gamma_i$, which are such that if $X_1, \ldots, X_{\beta_i}$ are individual variables and $P_1, \ldots, P_{\gamma_i}$ are statements, then $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ is a statement.

Note that since predicates are statements, one or more of $P_1, \ldots, P_{\gamma_i}$ may be predicates. As familiar examples of such $\prod_i$'s in the two-valued case, one could take $\prod_1(X, P)$ as $(X)P$ and $\prod_2(X, P)$ as $(\exists X)P$ if both the universal and existential quantifiers are basic. If $\prod_1(X, P)$ is the only basic quantifier, then $(\exists X)P$ would be defined as $\sim\prod_1(X, \sim P)$.

H 10. Each of $X_1, \ldots, X_{\beta_i}$ and $P_1, \ldots, P_{\gamma_i}$ occurs in $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$, and at least one occurrence of each X_j $(1 \leqslant j \leqslant \beta_i)$ does not occur as a part of the occurrence of a P_k $(1 \leqslant k \leqslant \gamma_i)$, and at least one occurrence of each P_k does not occur as part of the occurrences of any other P_k. Besides $X_1, \ldots, X_{\beta_i}$, no individual variables occur in $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ except those which occur as part of the P_k's.

H 11. If $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ is the same statement as $\prod_j(Y_1, \ldots, Y_{\beta_j}, Q_1, \ldots, Q_{\gamma_j})$, then $i = j$, X_1 is the same individual variable as $Y_1, \ldots, X_{\beta_i}$ is the same individual variable as Y_{β_i}, and P_1 is the same statement as $Q_1, \ldots, P_{\gamma_i}$ is the same statement as Q_{γ_j}.

Just as (X) "binds" the occurrences of X in P when the expression $(X)P$ is formed in two-valued logic, our generalized

quantifiers "bind" occurrences of variables in many-valued logic. This is indicated in our next assumption.

H 12. In each statement, the various occurrences of variables are classified as "free" or "bound". These concepts satisfy the following conditions:

(a) The free (bound) occurrences of individual variables in $F_i(P_1, \ldots, P_{a_i})$ are exactly those occurrences which are free (bound) in each of the $P_1, \ldots, P_{a_i}$ of $F_i(P_1, \ldots, P_{a_i})$.

(b) The occurrences of $X_1, \ldots, X_{\beta_i}$ are bound in $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$.

(c) For variables other than $X_1, \ldots, X_{\beta_i}$, the free (bound) occurrences in $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ are exactly those occurrences which are free (bound) in each of the parts $P_1, \ldots, P_{\gamma_i}$ of $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$.

To the best of our knowledge, no one has so far found need for quantifiers with either $\beta_i > 1$ or $\gamma_i > 1$. However, such quantifiers are distinctly possible, as may be seen from the following hypothetical example of a quantifier to which we will hereafter refer as $\prod_1(X_1, X_2, P_1, P_2)$.

Throughout our example, let P and Q be statements to which truth values are assigned by selecting values for $X, Y, Z_1, \ldots, Z_n$ from among the individuals of a given universe. For instance, P and Q might be predicates $F(X, Y, Z_1, \ldots, Z_n)$ and $G(X, Y, Z_1, \ldots, Z_n)$, or they might be predicates which depend on only some of $X, Y, Z_1, \ldots, Z_n$. Then, the truth value to be attached to $\prod_1(X, Y, P, Q)$ in any case is to be determined by the following prescriptions:

(1) $\prod_1(X, Y, P, Q)$ takes the truth value 1 for a given set of values for $Z_1, \ldots, Z_n$ if and only if there is a value of X such that for the given values of $Z_1, \ldots, Z_n$ and for all values of Y from the universe of individuals $F_1(P, Q)$ takes the truth value 2.

(2) $\prod_1(X, Y, P, Q)$ takes the truth value 2 for the given values of $Z_1, \ldots, Z_n$ if and only if for each value of X and Y from the universe of individuals and the given set of values for $Z_1, \ldots, Z_n$ we have that $P \& Q$ takes the truth value 1.

(3) $\prod_1(X, Y, P, Q)$ takes the truth value 3 for the given set of values for $Z_1, \ldots, Z_n$ if and only if for every value of X from

the universe of individuals and the given values of $Z_1, \ldots, Z_n$ there is a value of Y from the universe of individuals such that $F_1(P, Q)$ takes the truth value 3.

(4) $\prod_1(X, Y, P, Q)$ takes the truth value 4 for the given values of $Z_1, \ldots, Z_n$ in all cases except those stated in the prescriptions (1), (2), and (3).

(5) $\prod_1(X, Y, P, Q)$ never takes the truth values $5, \ldots, M$.

These prescriptions (1) to (5) are intuitive descriptions, and are intended purely for the reader's orientation. In their intuitive form they will play no part in our formal development. However, as we shall see, a formalized version of such intuitive descriptions will play an important role in our formal developments. In analogous fashion, from considerations of meaning, a student of many-valued logic might become interested in some form of quantification which he would describe to himself using prescriptions such as (1) to (5). He then would be faced with the problem of formalizing these intuitive descriptions.

As a more likely example, one might conceive of a kind of universal quantification $(X)F(X)$ which would be thought of as a generalized product $F(X_1)$ & $F(X_2)$ & $\ldots$, where $X_1, X_2, \ldots$ take on all possible values from a given universe of individuals. Then, using the truth function $\max(p_1, p_2)$ which is associated with P_1 & P_2, one could frame the following prescription for the truth values taken by $(X)P$, where P is a statement to which truth values are assigned by choosing values for $X, Z_1, \ldots, Z_n$ from the given universe of individuals, and where (X) binds X and only X in P:

$(X)P$ takes the truth value R for a given set of values for $Z_1, \ldots, Z_n$ from the universe of individuals if and only if for at least one value of X and for the given values of $Z_1, \ldots, Z_n$ the statement P takes the truth value R and for every value of X and for the given values of $Z_1, \ldots, Z_n$ the statement P takes no truth value which is greater than the truth value R.

In the case of the statement calculus, a predetermined notion of how the truth value of $F_i(P_1, \ldots, P_{a_i})$ depends upon the truth values of $P_1, \ldots, P_{a_i}$ was readily formalized as indicated in H 7 by simply assigning the appropriate truth function $f_i(p_1, \ldots, p_{a_i})$ to $F_i(P_1, \ldots, P_{a_i})$. We now face the problem of similarly translating

in a formal manner the intuitive truth-value prescriptions of our quantifiers. It is possible to manage a generalization of the method of assigning truth functions to statements which was used at the level of the statement calculus.[37] However, there is an alternative technique which appears to involve less complication and which will be called the method of "partial normal forms". This method will now be explained.

Each statement is to have M partial normal forms. Roughly speaking, the Rth ($1 \leqslant R \leqslant M$) partial normal form of a statement is a specification in terms of two-valued logic of the conditions under which the given statement takes the truth value R. For example, let $M = 3$ and consider the partial normal forms of a statement $F_1(P, Q)$, recalling that $f_1(p, q) = \max(1, q - p + 1)$:

(1) The first partial normal form is:

(q takes the value 1) $\vee$ ((q takes the value 2) & (p takes the value 2)) $\vee$ ((q takes the value 2) & (p takes the value 3)) $\vee$ ((q takes the value 3) & (p takes the value 3)).

(2) The second partial normal form is:

((q takes the value 2) & (p takes the value 1)) $\vee$ ((q takes the value 3) & (p takes the value 2)).

(3) The third partial normal form is:

((q takes the value 3) & (p takes the value 1)).

We see that (1), (2), and (3) merely state the conditions under which $f_1(p, q)$ takes the values 1, 2, and 3 respectively. Analogously, we can write partial normal forms for the statement $\prod_1(X, Y, P, Q)$. To this end, let p_k denote "p takes the value k", and correspondingly for q_k. We will conceive of p and q as truth functions whose values depend on at most the values of the variables $x, y, z_1, \ldots, z_n$. Now let $M = 4$ and recall the truth-value prescriptions for $\prod_1(X, Y, P, Q)$. We then have the following partial normal forms for $\prod_1(X, Y, P, Q)$:

(1) The first partial normal form is:

$$\sim (x) \sim (y) \{(q_2 \,\&\, p_1) \vee (q_3 \,\&\, p_2) \vee (q_4 \,\&\, p_3)\}.$$

[37] See Rosser and Turquette 1948.

(2) The second partial normal form is:

$$(x)(y)\ \{p_1 \ \&\ q_1\}.$$

(3) The third partial normal form is:

$$(x) \sim (y) \sim \{(q_3 \ \&\ p_1) \vee (q_4 \ \&\ p_2)\}.$$

(4) The fourth partial normal form is:

$\sim (N_1 \vee N_2 \vee N_3)$, where N_1, N_2, and N_3 are respectively the first, second, and third partial normal forms given by (1), (2), and (3).

Since no statement is to have two different truth values under precisely the same circumstances, its partial normal forms should be mutually exclusive. Also, since a statement should have at least one truth value, its partial normal forms should be exhaustive. This mutual exclusiveness and exhaustiveness is predicated on the assumption of a corresponding mutual exclusiveness and exhaustiveness of the conditions $p_1, \ldots, p_M$ and of the conditions $q_1, \ldots, q_M$. We shall express these requirements by means of the following hypothesis:

H 13. Let $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ be one of the basic quantifiers, and let $p_{j,k}$ $(1 \leqslant j \leqslant \gamma_i,\ 1 \leqslant k \leqslant M)$ be a two-valued predicate of $x_1, \ldots, x_{\beta_i}$, which is such that $p_{j,k} \neq p_{i,e}$ if either $k \neq e$ or $i \neq j$. We assume that there are two-valued statements N_r $(1 \leqslant r \leqslant M)$, which are built up out of the $p_{j,k}$ by means of the two-valued truth functions and quantifiers, and which have the following properties: Let A_j denote the two-valued logical product of all factors of the form $(x_1, \ldots, x_{\beta_i}) \sim (p_{j,k} \ \&\ p_{j,e})$ for which $k \neq e$. Let B_j denote the two-valued expression $(x_1, \ldots .x_{\beta_i})$ $(p_{j,1} \vee \ldots \vee p_{j,M})$. Let C denote the two-valued logical product of all factors of the form $\sim (N_r \ \&\ N_t)$ for which $r \neq t$. Let D denote the two-valued logical sum $(N_1 \vee \ldots \vee N_M)$. We shall assume that $((A_1 \ \&\ \ldots \ \&\ A_{\gamma_i}) \ \&\ (B_1 \ \&\ \ldots \ \&\ B_{\gamma_i})) \supset (C \ \&\ D)$ is provable in the ordinary two-valued predicate calculus.

Note that $p_{j,k}$ plays the role of the condition, (p_j takes the value k). Hence, in effect A_j and B_j say respectively that the conditions $p_{j,1}, \ldots, p_{j,M}$ are mutually exclusive and exhaustive. Likewise, C and D say that the conditions $N_1, \ldots, N_M$ are mutually exclusive and exhaustive respectively. We refer to $N_1, N_2, \ldots, N_M$

as the first, second, ..., Mth partial normal form which corresponds to the basic quantifier $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$.

In order to define a partial normal form N_r ($1 \leqslant r \leqslant M$) which corresponds to a function $F_i(P_1, \ldots, P_{\alpha_i})$, one can proceed as follows: Consider all the sets of values $p_1, \ldots, p_{\alpha_i}$ for which $f_i(p_1, \ldots, p_{\alpha_i}) = r$. For each such set construct the two-valued logical product p_{1,k_1} & p_{2,k_2} & ... & p_{α_i, k_c} where $k_i = p_i$ ($1 \leqslant i \leqslant c$ and $c = \alpha_i$). Then, our partial normal form N_r will be the two-valued logical sum of all such logical products.

For example, if $F_1(P_1, P_2)$ is a basic function and $M = 3$, then our three partial normal forms corresponding to $F_1(P_1, P_2)$ will be defined as follows:

$$N_1 \text{ is } (p_{1,1} \,\&\, p_{2,1}) \vee (p_{1,2} \,\&\, p_{2,1}) \vee (p_{1,3} \,\&\, p_{2,1}) \vee (p_{1,2} \,\&\, p_{2,2}) \vee (p_{1,3} \,\&\, p_{2,2}) \vee (p_{1,3} \,\&\, p_{2,3}).$$

$$N_2 \text{ is } (p_{1,1} \,\&\, p_{2,2}) \vee (p_{1,2} \,\&\, p_{2,3}).$$

$$N_3 \text{ is } p_{1,1} \,\&\, p_{2,3}.$$

A moment's reflection will indicate that N_1, N_2, and N_3 of the present example are in effect the same as the partial normal forms for $F_1(P, Q)$ which have already been described. We will now prove a theorem which will relate the definition of partial normal forms for our basic $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ with those for our functions $F_i(P_1, \ldots, P_{\alpha_i})$.

Theorem 4. 1. Consider the statement variables $p_{j,k}$ which are used in constructing the partial normal forms corresponding to $F_i(P_1, \ldots, P_{\alpha_i})$. It is assumed that these statement variables are such that $p_{j,k} \neq p_{i,e}$ if either $i \neq j$ or $k \neq e$. Let A_j denote the logical product of all factors $\sim (p_{j,k} \,\&\, p_{j,e})$ for which $k \neq e$, and let B_j denote $(p_{j,1} \vee \ldots \vee p_{j,M})$. That is, A_j and B_j are the same as in H 13 with the prefix $(x_1, \ldots, x_{\beta_i})$ omitted. We then have that $((A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_{\alpha_i}) \,\&\, (B_1 \,\&\, B_2 \,\&\, \ldots \,\&\, B_{\alpha_i})) \supset (C \,\&\, D)$ is provable in the two-valued statement calculus, where C and D are defined as in H 13 except that we now use the N_r's corresponding to $F_i(P_1, \ldots, P_{\alpha_i})$.

Proof. Note that in the two-valued statement calculus, the logical product $(B_1 \,\&\, B_2 \,\&\, \ldots \,\&\, B_{\alpha_i})$ is equivalent to the logical sum $(N_1 \vee N_2 \vee \ldots \vee N_M)$ and, thus, to D. Hence, our theorem will follow if we can prove $(A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_{\alpha_i}) \supset C$. For this

purpose assume that $(A_1 \& A_2 \& \ldots \& A_{\alpha_i})$ takes the truth value 1 and that C takes the truth value 2. It then follows from the definition of C that for some r and t such that $r \neq t$ there is a factor $\sim(N_r \& N_t)$ which takes the truth value 2. Hence, N_r and N_t both take the truth value 1. But by the definition of the partial normal form N_r, this means that there is at least one product $p_{1,k_1} \& p_{2,k_2} \& \ldots \& p_{\alpha_i,k_c}$ $(c = \alpha_i$ and $k_i = p_i)$ which takes the truth value 1. Hence, each factor of the product takes the truth value 1. Let p_{j,k_j} denote any such factor. Since $(A_1 \& \ldots \& A_{\alpha_i})$ takes the truth value 1, we know that for $k_j \neq d$ the factor $\sim(p_{j,k_j} \& p_{j,d})$ takes the truth value 1. Hence, since p_{j,k_j} takes the truth value 1, $p_{j,d}$ takes the truth value 2 for all $d \neq k_j$. But, since N_t is also 1, there is a product $p_{1,k_1^*} \& p_{2,k_2^*} \& \& \ldots \& p_{\alpha_i,k_c^*}$ such that each factor takes the truth value 1. Consider the factor p_{j,k_j^*}. By our previous result if $k_j^* \neq k_j$ we have a contradiction. But if $k_j^* = k_j$, the product $p_{1,k_1^*} \& \ldots \& p_{\alpha_i,k_c^*}$ would be the same as the product $p_{1,k_1} \& \ldots \& p_{\alpha_i,k_c}$ which contradicts the fact that $r \neq t$. Hence, we know that $(A_1 \& \ldots \& A_{\alpha_i}) \supset C$ always takes the truth value 1, and our theorem follows since the two-valued statement calculus is deductively complete.

We will now define a set of partial normal forms for any given statement by proceeding according to the following steps:

Step 1. Assign two-valued individual variables $x_1, x_2, \ldots$ to our given many-valued individual variables $X_1, X_2, \ldots$ respectively.

Step 2. Consider each statement P (if any) which satisfies the following conditions:

(I) P is not of either the form $F_i(P_1, \ldots, P_{\alpha_i})$ or the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$.

(II) P does not have two different free occurrences of the same free variable.

Let $Z_1, \ldots, Z_n$ denote all the individual variables (if any) which occur free in P. By step 1, the distinct two-valued variables $z_1, \ldots, z_n$ will be assigned to $Z_1, \ldots, Z_n$. Let $p_1(z_1, \ldots, z_n), \ldots, p_M(z_1, \ldots, z_n)$ be M distinct two-valued predicates of the distinct variables $z_1, \ldots, z_n$. We shall take $p_r(z_1, \ldots, z_n)$ $(1 \leqslant r \leqslant M)$ to be the rth partial normal form of P.

This choice of a partial normal form for P must be such that distinct P's will have completely distinct partial normal forms.

Moreover, if a statement Q satisfies conditions (I) and (II), and if $p_r(z_1, \ldots, z_n)$ is the rth partial normal form of P, then $p_r(y_1, \ldots, y_n)$ shall be the rth partial normal form of Q if and only if $Y_1, \ldots, Y_n$ are all the individual variables which occur free in Q, and furthermore Q is the result of replacing the $Z_1, \ldots, Z_n$ which occur free in P by $Y_1, \ldots, Y_n$ respectively.

Step 3. Consider each statement P (if any) which satisfies at least condition (I) of step 2. Let P^* be the result of replacing the first, second, ..., nth occurrence of a free variable (if any) in P by $Z_1, \ldots, Z_n$ respectively where $Z_1, \ldots, Z_n$ are distinct individual variables which do not occur in P. More precisely, counting the free occurrences of individual variables in P from left to right, the first occurrence of any free individual variable in P is replaced by Z_1. The second occurrence of any free individual variable in P is then replaced by Z_2, and in general the ith occurrence of any free individual variable in P is replaced by Z_i. After all the free occurrences of individual variables in P have been thus replaced by their proper Z_i's, the resulting statement is our P^*. Step 2 will give us a set of M partial normal forms for P^*. Let $X_1, \ldots, X_n$ be a set of individual variables (not necessarily distinct) such that P is the result of replacing the free occurrences of $Z_1, \ldots, Z_n$ in P^* by $X_1, \ldots, X_n$ respectively. If $p_r(z_1, \ldots, z_n)$ is the rth partial normal form of P^*, then take $p_r(x_1, \ldots, x_n)$ as the rth partial normal form of P where $x_1, \ldots, x_n$ are assigned to $X_1, \ldots, X_n$ by step 1.

Step 4. Consider any statement P (if any) which is either of the form $F_i(P_1, \ldots, P_{\alpha_i})$ or of the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$. We will assume that we have M partial normal forms for each P_j $(1 \leqslant j \leqslant \alpha_i)$ and each P_h^* $(1 \leqslant h \leqslant \gamma_i)$. Directions will be given for constructing the M partial normal forms of P from the assumed partial normal forms of P_j and P_h^*. In view of our previous steps for constructing partial normal forms, this will give directions for constructing partial normal forms for any statement. Hence, let $p_{j,m}^0$ be the mth partial normal form of P_j, and let $p_{h,e}^*$ be the eth partial normal form of P_h^*. If P is of the form $F_i(P_1, \ldots, P_{\alpha_i})$, then its rth partial normal form will be the N_r constructed according to the definition of partial normal forms for the functions $F_i(P_1, \ldots, P_{\alpha_i})$ except that now we will use the partial normal form p_{j,k_j}^0 in the

place of the statement variable p_{j,k_j}. If P is of the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$, then its rth partial normal form will be N_r as defined in H 13 except that we will use $p_{h,e}^*$ in the place of $p_{j,k}$ when $h = j$ and $k = e$.

We can now define "the fundamental constituents" from which the partial normal forms of a statement P are constructed by using two-valued truth functions and quantifiers:

Let P_j be a statement which occurs as a part of P, and which satisfies at least condition (I) of step 2. Also, let P_j^* be related to P_j as P^* is related to P in step 3. Let $p_{j,r}^*$ be the rth partial normal form of P_j^*. Now list all the $p_{j,r}^*$'s corresponding to all the P_j^*'s which are formed from all the P_j's, satisfying our conditions, which occur as parts of P. If from this list of $p_{j,r}^*$'s we now eliminate all of those which are either exact duplicates or which differ only by a change of argument variables, then the remaining set of $p_{j,r}^*$'s is our desired set of fundamental constituents of the partial normal forms of a statement P.

If we let $p_{a,b}^*$ temporarily denote a fundamental constituent of a partial normal form of a statement P, then we can state and prove the following theorem:

Theorem 4. 2. Suppose we have given any statement P and all of its fundamental constituents $p_{a,b}^*$. Let A_j denote the two-valued logical product of all factors of the form $(z_1, \ldots, z_{n_j}) \sim \sim (p_{j,k}^* \ \& \ p_{j,e}^*)$ for $k \neq e$ and where $z_1, \ldots, z_{n_j}$ are the argument variables of $p_{j,k}^*$. Let B_j denote the two-valued expression $(z_1, \ldots, z_{n_j}) \ (p_{j,1}^* \vee \therefore \vee p_{j,M}^*)$. Let Q denote the two-valued expression $(A_1 \ \& \ \ldots \ \& \ A_m) \ \& \ (B_1 \ \& \ \ldots \ \& \ B_m)$ where m is the number of P_j's which occur as parts of P as specified in the definition of a fundamental constituent of a partial normal form of P. Let C denote the two-valued logical product of all factors of the form $\sim (N_r \ \& \ N_t)$ for $r \neq t$ and where N_r is the rth partial normal form of P. Finally, let D denote the two-valued logical sum of the partial normal forms of P, namely, $N_1 \vee \ldots \vee N_M$. We then have $Q \supset (C \ \& \ D)$ provable in the two-valued predicate calculus.

Proof. Use induction on the structure of P.

(a) If P is not either of the form $F_i(P_1, \ldots, P_{\alpha_i})$ or of the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$, then it satisfies condition (I) of step 2. Hence, P is the same as P_j, and Q is $A_j \ \& \ B_j$. Also, by

step 3, C is A_j^* where A_j^* is the same as A_j except that the prefix $(z_1, \ldots, z_{n_j})$ has been omitted. Likewise, D is B_j^* which is the same as B_j except that the prefix $(z_1, \ldots, z_{n_j})$ is omitted. But $A_j \supset A_j^*$ and $B_j \supset B_j^*$ are provable in the two-valued predicate calculus. Hence, so is $Q \supset (C \;\&\; D)$.

(β) If P is either of the form $F_i(P_1, \ldots, P_{a_i})$ or of the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$, then we will assume that our theorem holds for the statements $P_1, \ldots, P_{a_i}$ and $P_1^*, \ldots, P_{\gamma_i}^*$. Under this assumption, we will then show that our theorem must hold for P.

Let Q_a, C_a, D_a, and $N_{r,a}$ be related to P_a $(1 \leqslant a \leqslant a_i)$ as Q, C, D, and N_r respectively are related to P in the statement of the present theorem. Likewise, for the relationship of Q_a^*, C_a^*, D_a^*, and $N_{r,a}^*$ to P_a^* when $1 \leqslant a \leqslant \gamma_i$. By our assumption of induction we then have,

(1) $Q_a \supset (C_a \;\&\; D_a)$ when $1 \leqslant a \leqslant a_i$.

(2) $Q_a^* \supset (C_a^* \;\&\; D_a^*)$ when $1 \leqslant a \leqslant \gamma_i$.

But from the definition of Q, Q_a, and Q_a^* it is clear that $Q \supset Q_a$ and $Q \supset Q_a^*$ are provable in the two-valued predicate calculus. Hence, this result with (1) and (2) will give,

(3) $Q \supset (C_a \;\&\; D_a)$ and $Q \supset (C_a^* \;\&\; D_a^*)$

are both provable in the two-valued predicate calculus.

Now consider the rth partial normal form of P.

Case 1. If P is of the form $F_i(P_1, \ldots, P_{a_i})$, then the rth partial normal form N_r of P is constructed from the rth partial normal form of the function $F_i(P_1, \ldots, P_{a_i})$ by replacing p_{j,k_j} by $N_{k_j,j}$. But theorem 4. 1 would continue to hold after such a replacement of p_{j,k_j} by $N_{k_j,j}$. Note, however, that after this replacement the expression $(A_1 \;\&\; \ldots \;\&\; A_{a_i}) \;\&\; (B_1 \;\&\; \ldots \;\&\; B_{a_i})$ is just the same as $C_a \;\&\; D_a$. Hence, we have $(C_a \;\&\; D_a) \supset (C \;\&\; D)$. Using (3), it follows that $Q \supset (C \;\&\; D)$ is provable in the two-valued predicate calculus, and we have our theorem.

Case 2. If P is of the form $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$, then the rth partial normal form N_r of P is constructed from the rth partial normal form of the basic function $\prod_i(X_1, \ldots, X_{\beta_i}, P_1^*, \ldots, P_{\gamma_i}^*)$ by replacing $p_{j,k}$ by $N_{k,j}^*$. But H 13 will continue to hold after such a replacement of $p_{j,k}$ by $N_{k,j}^*$. Let $((A_1 \;\&\; \ldots \;\&\; A_{\gamma_i}) \;\&\; (B_1 \;\&\; \ldots \;\&\; B_{\gamma_i}))^0$ denote the result of replacing $p_{j,k}$'s in the

expression $(A_1 \& \ldots \& A_{\gamma_i}) \& (B_1 \& \ldots \& B_{\gamma_i})$ of H 13 by $N^*_{k,j}$'s. Since H 13 continues to hold, we have $((A_1 \& \ldots \& A_{\gamma_i}) \& (B_1 \& \ldots \& B_{\gamma_i}))^0 \supset (C \& D)$ provable in the two-valued predicate calculus, where $C \& D$ is as defined in the present theorem. Also, one can get $(x_1, \ldots, x_n)(C^*_a \& D^*_a) \supset ((A_1 \& \ldots \& A_{\gamma_i}) \& (B_1 \& \ldots \& \& B_{\gamma_i}))^0$ provable in the two-valued predicate calculus. But since none of $x_1, \ldots, x_n$ occur free in Q, one can use (3) to get $Q \supset (x_1, \ldots, x_n)\ (C^*_a \& D^*_a)$ provable in the two-valued predicate calculus. Hence, this with our previous results gives $Q \supset (C \& D)$, and we have our present theorem.

We are now in a position to define a "truth-value stipulation" for acceptable statements which are built up using the functions $F_i(P_1, \ldots, P_{\alpha_i})$ and the basic quantifiers $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. In effect, we want such a statement to be acceptable if and only if it takes one of the truth values $1, \ldots, S$. This is accomplished as follows:

Let P be any statement which is built up using our basic functions and quantifiers. Let Q be constructed out of the fundamental constituents of the partial normal forms of P as indicated in theorem 4. 2. We will then say that P is acceptable according to our truth-value stipulation if and only if the expression $Q \supset (N_1 \vee N_2 \vee \ldots \vee N_S)$ is provable in the two-valued predicate calculus.

With the class of acceptable statements determined by a truth-value stipulation thus defined, we can proceed as before to discuss the relationship between this class of acceptable statements and that which is defined by another stipulation. In the next chapter we will give an axiomatic stipulation of acceptable statements which are built up using $F_i(P_1, \ldots, P_{\alpha_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. Hence, after such an axiomatic stipulation is given, we can examine questions of the plausibility of our axiomatic stipulation relative to a truth-value stipulation, questions of the deductive completeness of an axiomatic stipulation relative to a truth-value stipulation, and all such related questions, with no essential change in the notions of plausibility, deductive completeness, and related concepts that were used in the previous chapter.

However, before turning to such a study, it will be convenient

to broaden our definition of standard conditions for many-valued functions so that it will cover the case of a many-valued operator (X) which is in many respects analogous to the two-valued universal quantifier. This will be accomplished by defining standard conditions for the partial normal forms of many-valued expressions of the form $(X)P$. The following is such a definition:

Suppose that P has only free occurrences of the individual variable Z, and let $p_r(z)$ denote the rth partial normal form of P as defined in step 3 of the definition of partial normal forms for any statement. Then the partial normal forms N_r of $(X)P$ satisfy standard conditions if and only if they are such that the following expression is provable in the ordinary two-valued predicate calculus,

$$(z)\ (p_1(z) \vee \ldots \vee p_S(z)) \equiv (N_1 \vee \ldots \vee N_S).$$

It should be remarked that this definition, as with our earlier definition of standard conditions, is essentially independent of our axiomatization of many-valued predicate calculi.

It will be observed that if the partial normal forms of $(X)P$ satisfy standard conditions, then $(X)P$ is acceptable according to our truth-value stipulation if and only if P is always acceptable by our truth-value stipulation. This state of affairs is sometimes expressed by saying that $(X)P$ takes a designated truth value if and only if P always takes a designated truth value. [38]

With our broadened definition of standard conditions given, we are now in a position to consider the problem of axiomatizing many-valued predicate calculi, that is of giving an axiomatic stipulation for acceptable statements which may now contain predicates and quantifiers $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. The chief task of the next chapter is to give such an axiomatic stipulation and to compare it with truth-value stipulations.

[38] Rosser and Turquette 1948.

V

AXIOMATIZATION OF MANY-VALUED PREDICATE CALCULI

As in the case of statement calculi, a finite number of axiom schemes will be used to define our axioms. Since we are now concerned with predicate calculi, it is not enough to proceed as in the case of statement calculi and list axiom schemes which merely indicate how acceptable statements are formed from given statements by means of statement operators. In addition to this, our axiom schemes must now indicate how acceptable statements are formed from given statements, which may include predicates, by means of statement operators and quantifiers. This requires the use of both statement and individual variables in the description of our axiom schemes as well as other complexities which will be indicated as we list our axiom schemes.

Our list of axiom schemes for many-valued predicate calculi will be obtained as follows:

(a) We will retain A 1 to A 7 of chapter III with the understanding that a statement may now contain predicates and quantifiers.

(b) To the axiom schemes obtained by (a) we will add A 8 to A 10 which are to be listed below, but our descriptions of these new axiom schemes A 8 to A 10 will be considerably simplified if we first introduce some additional notation. In particular, we will use the notation $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ to refer to a many-valued statement which is closely analogous to, and, in fact, is constructed after the pattern of the rth partial normal form of the statement $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. If N_r is the rth partial normal form of $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$, then $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ will be constructed according to the following plan:

Step 1. Rewrite N_r using only the two-valued operators $\sim$, $\supset$, and (x), (y), ... This is possible by H 13. Call the result N_r^*.

Note that N_r^* is built up in a well-defined manner out of the $p_{j,k}$'s of H 13 by means of the two-valued $\sim$, $\supset$, (x), (y), ...

Since by (a) we are retaining A 1 to A 7, it is already assumed that a choice of functions defined in terms of our $F_i(P_1, \ldots, P_{a_i})$ has been made for the $\supset$ and $J_k(\ )$ $(1 \leqslant k \leqslant M)$ which appear in the axiom schemes. Now, in addition to this choice, we will select a many-valued $\sim$ defined in terms of the functions $F_i(P_1, \ldots, P_{a_i})$, and a many-valued (X) defined in terms of the $F_i(P_1, \ldots, P_{a_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. We then can take the next step.

Step 2. Using the many-valued $\sim$, $\supset$, (X), and $J_k(\ )$ which have been selected, construct $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ out of $J_k(P_j)$ $(1 \leqslant j \leqslant \gamma_i$ and $1 \leqslant k \leqslant M)$ using our many-valued $\sim$, $\supset$, (X), (Y), ... in exactly the same well-defined manner that N_r^* is constructed out of $p_{j,k}$ using the two-valued $\sim$, $\supset$, (x), (y), ...

As an illustration of this new notation, consider the partial normal forms of $\prod_1(X, Y, P, Q)$ that were listed in the last chapter. By step 1, we could take $\mathbf{v}$ and &, which appear in expressions of the form $P \mathbf{v} Q$ and $P \mathbin{\&} Q$ in the statement of the partial normal forms of $\prod_1(X, Y, P, Q)$, as abbreviations for the two-valued expressions $\sim(P) \supset Q$, and $\sim(P \supset \sim Q)$ respectively. Suppose that we continue to use this same method of abbreviation when $\sim$ and $\supset$ are our chosen many-valued operators. For $M = 4$, we could then write our four $\mathbf{n}_r(\prod_1(X, Y, P, Q))$ as follows:

$$\mathbf{n}_1(\textstyle\prod_1(X, Y, P, Q)) \text{ is } \sim(X) \sim(Y) \{(J_2(Q) \mathbin{\&} J_1(P)) \mathbf{v} (J_3(Q) \mathbin{\&} J_2(P)) \mathbf{v} (J_4(Q) \mathbin{\&} J_3(P))\}.$$

$$\mathbf{n}_2(\textstyle\prod_1(X, Y, P, Q)) \text{ is } (X)(Y)\{J_1(P) \mathbin{\&} J_1(Q)\}.$$

$$\mathbf{n}_3(\textstyle\prod_1(X, Y, P, Q)) \text{ is } (X) \sim(Y) \sim\{(J_3(Q) \mathbin{\&} J_1(P)) \mathbf{v} (J_4(Q) \mathbin{\&} J_2(P))\}.$$

$$\mathbf{n}_4(\textstyle\prod_1(X, Y, P, Q)) \text{ is } \sim(\mathbf{n}_1(\prod_1(X, Y, P, Q)) \mathbf{v} \mathbf{n}_2(\prod_1(X, Y, P, Q)) \mathbf{v} \mathbf{n}_3(\prod_1(X, Y, P, Q))).$$

We are now in a position to list our axiom schemes A 8 to A 10, and as in the case of A 1 to A 7 we recall that each axiom scheme defines a set of axioms.

A 8. $(X)P \supset Q$, where P is a statement, X and Y are individual variables, Q is the result of replacing each free occurrence (if any) of X in P by Y, and no bound occurrence of Y (if any) in Q is the result of replacing a free occurrence of X in P by Y.

A 9. $(X)(P \supset Q) \supset (P \supset (X)Q)$, where P is a statement containing no free occurrences of X, X is an individual variable, and Q is a statement.

A 10. $n_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \supset J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$, where $1 \leqslant r \leqslant M$, $1 \leqslant i \leqslant c$, and $P_1, \ldots, P_{\gamma_i}$ are statements, and $X_1, \ldots, X_{\beta_i}$ are individual variables.

Note that A 10 is really a set of cM axiom schemes. For example, if $M = 4$ and $\prod_1(X, Y, P, Q)$ is our basic quantifier, then A 10 would generate the following axiom schemes:

(1) $n_1(\prod_1(X, Y, P, Q)) \supset J_1(\prod_1(X, Y, P, Q))$
(2) $n_2(\prod_1(X, Y, P, Q)) \supset J_2(\prod_1(X, Y, P, Q))$
(3) $n_3(\prod_1(X, Y, P, Q)) \supset J_3(\prod_1(X, Y, P, Q))$
(4) $n_4(\prod_1(X, Y, P, Q)) \supset J_4(\prod_1(X, Y, P, Q))$

If we now give a list of rules of inference, our axiomatic stipulation will be defined. Such a list will be specified as follows:

(a) We will retain R 1 with the understanding concerning our extended conception of a statement.

(b) To R 1 there will be added an R 2 as follows:

R 2. If P is any acceptable statement, then $(X)P$ is an acceptable statement.

The axiom schemes A 1 to A 10, and rules R 1 and R 2 define our axiomatic stipulation of acceptable statements in a many-valued predicate calculus. It is now possible to study the relation of the class of acceptable statements given by our axiomatic stipulation to the class of acceptable statements defined by a truth-value stipulation as specified in the previous chapter. To this end, it will be convenient to extend our definition of "$P_1, \ldots, P_n \vdash Q$" to allow the use of axioms A 1 to A 10 and rules R 1 and R 2. Clearly, this will present no difficulty as was remarked when the yields sign was first defined.

It will be useful, also, to define what we shall call "the rth partial normal form of an axiom scheme". In defining this concept, we will give a series of definitions by the following cases:

Case 1. Consider each of the axiom schemes A 1 to A 7. Such an axiom scheme may be taken as a statement which is built up out of P, Q, R, etc. using the statement functions $F_i(P_1, \ldots, P_{a_i})$. The rth partial normal form of this statement will be taken as the

rth partial normal form of the given one of the axiom schemes A 1 to A 7.

For example, if $M = 3$ and $F_1(P, Q)$ is our chosen $P \supset Q$, then the three partial normal forms of A 1 will be as follows:

N_1 is $(p_1 \,\&\, q_1) \vee (p_1 \,\&\, q_2) \vee (p_1 \,\&\, q_3) \vee$
$(p_2 \,\&\, q_1) \vee (p_2 \,\&\, q_2) \vee (p_2 \,\&\, q_3) \vee$
$(p_3 \,\&\, q_1) \vee (p_3 \,\&\, q_2) \vee (p_3 \,\&\, q_3)$.
N_2 is $p_1 \,\&\, \sim p_1$.
N_3 is $q_1 \,\&\, \sim q_1$.

In this particular example, note that since $((p_1 \vee p_2 \vee p_3) \,\&\, (q_1 \vee \vee q_2 \vee q_3)) \supset N_1$, the axiom scheme A 1 is acceptable according to our truth-value stipulation.

Case 2. Consider the axiom scheme A 8. If $P(X)$ is related to $P(Y)$ as P is related to Q in A 8, then A 8 may be considered as a statement which is built up out of $P(X)$ and $P(Y)$ using our functions $F_i(P_1, \ldots, P_{\alpha_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. The rth partial normal form of this statement will be chosen as the rth partial normal form of the axiom scheme A 8.

Henceforth, let $N_r(\_\_)$ denote the rth partial normal form of __, and consider the problem of constructing a particular $N_r(\text{A } 8)$. To this end take $M = 3$ and $F_1(P, Q)$ as $P \supset Q$, and suppose that we have given $N_r((X)P)$ which is built up out of $p_k(x)$'s using the two-valued $\supset$, (x), and $\sim$, where $p_k(x)$ is related to $p_k(y)$ as $P(X)$ is related to $P(Y)$ in case 2. We then can write the three partial normal forms of A 8 as follows:

$N_1(\text{A } 8)$ is $(N_1((X)P) \,\&\, p_1(y)) \vee (N_2((X)P) \,\&\, p_1(y)) \vee$
$(N_3((X)P) \,\&\, p_1(y)) \vee (N_2((X)P) \,\&\, p_2(y)) \vee$
$(N_3((X)P) \,\&\, p_2(y)) \vee (N_3((X)P) \,\&\, p_3(y))$.
$N_2(\text{A } 8)$ is $(N_1((X)P) \,\&\, p_2(y)) \vee (N_2((X)P) \,\&\, p_3(y))$.
$N_3(\text{A } 8)$ is $N_1((X)P) \,\&\, p_3(y)$.

In this particular case, if $S = 2$ and $N_r((X)P)$ satisfies standard conditions, then A 8 is acceptable according to our truth-value stipulation. This can be proved as follows:

Henceforth, let $Q(\_\_)$ denote the Q corresponding to __ as defined in theorem 4. 2. We wish to show that $Q(\text{A } 8) \supset (N_1(\text{A } 8) \vee N_2(\text{A } 8))$ is provable in the two-valued predicate calculus, where $Q(\text{A } 8)$ is built up out of our $p_k(x)$'s. Assume $Q(\text{A } 8) \,\&\, \sim (N_1(\text{A } 8) \vee N_2(\text{A } 8))$

and deduce a contradiction. Using theorem 4. 2 with our assumption, we can get N_3(A 8). Since this is $N_1((X)P)$ & $p_3(y)$, we have $N_1((X)P)$ and $p_3(y)$.

Hence, we can get $N_1((X)P) \vee N_2((X)P)$. Since $N_r((X)P)$ satisfies standard conditions, we have $(x)(p_1(x) \vee p_2(x))$. Thus, we get $p_1(y) \vee p_2(y)$, and with $p_3(y)$ we have $(p_1(y) \vee p_2(y))$ & $p_3(y)$. But this latter result is the same as $\sim(\sim(p_1(y)\ \&\ p_3(y))\ \&\ \sim(p_2(y)\ \&\ p_3(y)))$ which contradicts Q(A 8).

By a similar procedure, one can illustrate the partial normal forms for A 9 and A 10 which will be defined by our next two cases, and then test the acceptability of these axiom schemes according to our truth-value stipulation. Hence, we will merely state our definitions for the case of A 9 and A 10, and no attempt will be made to illustrate these definitions.

Case 3. Consider axiom scheme A 9. If P and $Q(X)$ are as the P and Q of A 9, then A 9 may be considered as a statement which is built up out of P and $Q(X)$ using the functions $F_i(P_1, \ldots, P_{\alpha_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. The rth partial normal form of this statement will be taken as the rth partial normal form of the axiom scheme A 9.

Case 4. Consider the axiom scheme A 10. If $\boldsymbol{n}_r(\ )$ is as defined for A 10, then A 10 may be considered as a statement which is built up out of $P_1, \ldots, P_{\gamma_i}$ using the functions $F_i(P_1, \ldots, P_{\alpha_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. The rth partial normal form of this statement will be taken as the rth partial normal form of the axiom scheme A 10.

This concludes our definition of the rth partial normal form of an axiom scheme, and in terms of such partial normal forms one can test the acceptability of our axiom schemes A 1 to A 10 using the same criterion that defines the acceptability of statements according to our truth-value stipulation. In what follows, such a criterion will be used when we speak of the acceptability of any one of our axiom schemes A 1 to A 10 according to our truth-value stipulation.

Lemma 5. 1. 1. If W is a statement which is an instance of an axiom scheme A 1 to A 10 that is acceptable by our truth-value stipulation, then the statement W is acceptable according to our truth-value stipulation.

Proof. We will illustrate the proof using A 1 and A 8. The reader will find it instructive to complete the proof for the other axioms.

By definition, $N_r(\text{A } 1)$ is built up out of p_k and q_k using the two-valued $\sim$ and $\supset$. Since A 1 is acceptable by our truth-value stipulation, we have $Q(\text{A } 1) \supset (N_1(\text{A } 1) \vee \ldots \vee N_S(\text{A } 1))$ provable in the two-valued predicate calculus where $Q(\text{A } 1)$ is built up out of p_k and q_k. Since W is an instance of A 1, there are statements P^* and Q^* such that W is $Q^* \supset (P^* \supset Q^*)$. By the definition of partial normal forms for any statement, $N_r(W)$ results from $N_r(\text{A } 1)$ by replacing the p_k and q_k of $N_r(\text{A } 1)$ by $N_k(P^*)$ and $N_k(Q^*)$ respectively. Let Q_W denote the result of replacing p_k and q_k in $Q(\text{A } 1)$ by $N_k(P^*)$ and $N_k(Q^*)$ respectively. Note that Q_W is just C & D for W as defined in theorem 4. 2. Since an instance of a provable formula of the two-valued predicate calculus is provable, $Q_W \supset (N_1(W) \vee \ldots \vee N_S(W))$ is provable in the two-valued predicate calculus. Since Q_W is C & D, we have $Q(W) \supset Q_W$ by theorem 4. 2. Hence, $Q(W) \supset (N_1(W) \vee \ldots \vee N_S(W))$ is provable in the two-valued predicate calculus, and our lemma holds when W is an instance of A 1.

By definition, $N_r(\text{A } 8)$ is built up out of $p_k(x)$'s using the two-valued $\sim$, $\supset$, and (x). Since A 8 is acceptable by our truth-value stipulation, we have $Q(\text{A } 8) \supset (N_1(\text{A } 8) \vee \ldots \vee N_S(\text{A } 8))$ provable in the two-valued predicate calculus where $Q(\text{A } 8)$ is built up out of $p_k(x)$'s. Since W is an instance of A 8, there are statements P^* and Q^* such that W is $(X)P^* \supset Q^*$ where Q^* is related to P^* as Q is related to P in A 8. By the definition of partial normal forms for any statement, $N_r(W)$ results from $N_r(\text{A } 8)$ by replacing $p_k(x)$ by $N_k(P^*)$. Let Q^*_W denote the result of replacing $p_k(x)$ in $Q(\text{A } 8)$ by $N_k(P^*)$. Since an instance of a provable formula is provable, we have $Q^*_W \supset (N_1(W) \vee \ldots \vee N_S(W))$ provable in the two-valued predicate calculus. Now assume $Q(W)$. Since $Q(W)$ is the same as $Q(P^*)$ except perhaps for a change of bound variables, by theorem 4. 2 we then have C & D for P^*. But the structure of C & D for P^* is such that by quantification we can get our Q^*_W. Since there are no free variables in $Q(W)$, we can use the deduction theorem to get $Q(W) \supset Q^*_W$ provable in the two-valued predicate calculus. Hence, $Q(W) \supset (N_1(W) \vee \ldots \vee N_S(W))$ is provable, and our lemma follows for A 8.

We will now extend our definition of plausible functions to cover the case of our new axiom schemes and rules. The functions $j_k(p)$, $\sim(p)$, $N_r((X)P)$ and $\supset(p, q)$ will be called "plausible" if they satisfy the following conditions:

(1) The functions are such that the axiom schemes A 1 to A 10 are acceptable by our truth-value stipulation.

(2) The function $\supset(p, q)$ is such that if P is acceptable by our truth-value stipulation and Q is not acceptable, then $P \supset Q$ is not acceptable by our truth-value stipulation.

(3) The function $N_r((X)P)$ is such that if P is acceptable according to our truth-value stipulation, then so is $(X)P$.

Lemma 5. 1. 2. If the functions $j_k(p)$, $\sim(p)$, $N_r((X)P)$, and $\supset(p, q)$ are plausible, and if W is a statement which is such that $\vdash W$, then W is acceptable according to our truth-value stipulation.

Proof. If W is an instance of an axiom scheme, then our lemma follows by (1) of the definition of plausible functions and lemma 5. 1. 1. If R 1 or R 2 are used to obtain $\vdash W$, then our lemma will follow from (2) and (3) of the definition of plausible functions. A precise treatment of this proof will require an inductive argument on the number of steps in a demonstration of $\vdash W$. The reader will find it easy to construct such a proof.

Lemma 5. 1. 3. Let W^* denote a two-valued expression which is built up out of the fundamental constituents $p_{j,k}$ of $N_r(P_j)$ using the two-valued operators $\sim$, (x), and $\supset$. Let W be built up out of the constituents $J_k(P_j)$ using the many-valued operators $\sim$, (X), and $\supset$ in the same well-defined manner that W^* is built up out of $p_{j,k}$'s using the two-valued $\sim$, (x), and $\supset$. If $Q(W)$ is as defined in theorem 4. 2 and $j_k(p)$, $\sim(p)$, $N_r((X)P)$, and $\supset(p, q)$ satisfy standard conditions, then $Q(W) \supset ((\sum_1^s N_i(W)) \equiv W^*)$ is provable in the two-valued predicate calculus.

Proof. Use induction on the structure of W.

(a) If W is a $J_k(P_j)$, then W^* is $p_{j,k}$. Also, since $j_k(p)$ satisfies standard conditions, by the definition of partial normal forms for statements built up using $F_i(P_1, \ldots, P_{a_i})$, the logical sum $\sum_1^s N_i(J_k(P_j))$ is such that one term is $p_{j,k}$ and all other terms are of the form $p \,\&\, \sim p$. Hence, we have $\sum_1^s N_i(J_k(P_j)) \equiv p_{j,k}$, and so $Q(W) \supset (\sum_1^s N_i(W) \equiv W^*)$ is provable in the two-valued predicate calculus.

(β) Under the assumptions of our lemma, assume $Q(W) \supset \supset (\sum_1^s N_i(W) \equiv W^*)$ for all W's with less than n symbols and prove that we have $Q(W) \supset (\sum_1^s N_i(W) \equiv W^*)$ for W with n symbols. Let W have n symbols and consider the following cases.

Case 1. If W is of the form $\sim H$, then $\sum_1^s N_i(W)$ is $\sum_1^s N_i(\sim H)$. Since $\sim(p)$ satisfies standard conditions, by the definition of partial normal forms for statements built up using $F_i(P_1, \ldots, P_{\alpha_i})$ we know that $\sum_1^s N_i(\sim H) \equiv \sum_{S+1}^M N_i(H)$. However, if we assume $Q(W)$, by theorem 4. 2 we can get $\sum_{S+1}^M N_i(H) \equiv \sim \sum_1^s N_i(H)$, and so $\sum_1^s N_i(\sim H) \equiv \sim \sum_1^s N_i(H)$. By assumption ($\beta$), $\sum_1^s N_i(H) \equiv H^*$ and so we have $\sim \sum_1^s N_i(H) \equiv \sim H^*$. Hence, $\sum_1^s N_i(\sim H) \equiv \sim H^*$, which is just $\sum_1^s N_i(W) \equiv W^*$. Now use the deduction theorem.

Case 2. If W is of the form $(X)H$, then $\sum_1^s N_i(W)$ is $\sum_1^s N_i((X)H)$. Since $N_r((X)P)$ satisfies standard conditions, we have $\sum_1^s N_i((X)H) \equiv \equiv (x) \sum_1^s N_i(H)$. If we assume $Q(W)$, by assumption (β) we have $\sum_1^s N_i(H) \equiv H^*$. We thus can get $(x) \sum_1^s N_i(H) \equiv (x)H^*$. Hence, we have $\sum_1^s N_i((X)H) \equiv (x)H^*$ which is just $\sum_1^s N_i(W) \equiv W^*$. Now use the deduction theorem.

Case 3. If W is of the form $G \supset H$, then $\sum_1^s N_i(W)$ is $\sum_1^s N_i(G \supset H)$. Since $\supset(p, q)$ satisfies standard conditions, by the definition of partial normal forms for statements built up using $F_i(P_1, \ldots, P_{\alpha_i})$ we know that $\sum_1^s N_i(G \supset H) \equiv (\sum_{S+1}^M N_i(G) \vee \sum_1^s N_i(H))$.

Now assume $Q(W)$. By theorem 4. 2 we can then get $\sum_{S+1}^M N_i(G) \equiv \equiv \sim \sum_1^s N_i(G)$. Thus, $\sum_1^s N_i(G \supset H) \equiv \sim \sum_1^s N_i(G) \vee \sum_1^s N_i(H)$ or $\sum_1^s N_i(G \supset H) \equiv \sum_1^s N_i(G) \supset \sum_1^s N_i(H)$. But by assumption ($\beta$), we have $\sum_1^s N_i(G) \equiv G^*$ and $\sum_1^s N_i(H) \equiv H^*$. Hence, $\sum_1^s N_i(G \supset H) \equiv \equiv (G^* \supset H^*)$ which is just $\sum_1^s N_i(W) \equiv W^*$. Now use the deduction theorem.

Since we have exhausted the structure of W our lemma follows.

Theorem 5. 1. If the functions $j_k(p)$, $\sim(p)$, $N_r((X)P)$, and $\supset(p, q)$ each satisfies standard conditions, and W is a many-valued statement such that $\vdash W$, then W is acceptable according to our truth-value stipulation.

Proof. We shall show that since our functions satisfy standard conditions, they are plausible. Hence, our theorem follows by lemma 5. 1. 2. Our proof is by cases.

Case 1. If condition (1) of the definition of plausible functions, is to be satisfied, then we must show that A 1 to A 7 are acceptable

by our truth-value stipulation, and likewise for A 8 to A 10.

Subcase 1. If we consider A 1 to A 7, then use must be made of the definition of partial normal forms for statements built up by means of $F_i(P_1, \ldots, P_{a_i})$. However, since our functions satisfy standard conditions, a truth-table check will indicate that our axioms A 1 to A 7 always take designated truth-values. This means that the B_j of theorem 4. 2 for each axiom scheme is such that we can verify by two-valued truth-tables that the following formula takes the value "truth" exclusively and, hence, is provable in the ordinary two-valued predicate calculus.

(1) $(B_1 \,\&\, \ldots \,\&\, B_m) \supset (N_1 \vee \ldots \vee N_S)$, where N_r $(1 \leqslant r \leqslant S)$ is the rth partial normal form of the given axiom scheme. Hence, by (1) we have $Q \supset (N_1 \vee \ldots \vee N_S)$, where Q is as defined in theorem 4. 2 for the given axiom scheme. Thus, A 1 to A 7 are acceptable by our truth-value stipulation.

Subcase 2. First consider A 8. We wish to show that $Q(\text{A } 8) \supset \supset (N_1(\text{A } 8) \vee \ldots \vee N_S(\text{A } 8))$ is provable in the two-valued predicate calculus. Assume $Q(\text{A } 8) \,\&\, \sim (N_1(\text{A } 8) \vee \ldots \vee N_S(\text{A } 8))$ and deduce a contradiction. By theorem 4. 2, we can get $N_{S+1}(\text{A } 8) \vee \ldots \vee N_M(\text{A} 8)$. Since A 8 is of the form $R \supset S$, by the definition of partial normal forms for statements built up using $F_i(P_1, \ldots, P_{a_i})$ and the fact that $\supset (r, s)$ satisfies standard conditions, we know that one can thus infer a logical sum of terms of the form $N_D((X)P(X)) \,\&\, \& \, N_U(P(Y))$ where D denotes a designated value and U an undesignated value. Let $N_D((X)P(X)) \,\&\, N_U(P(Y))$ denote any one of the terms of this sum. From such a term one can infer $(N_1((X)P(X)) \vee \ldots \vee N_S((X)P(X))) \,\&\, (N_{S+1}(P(Y)) \vee \ldots \vee N_M(P(Y)))$. But since $N_r((X)P)$ satisfies standard conditions, we can then get $(N_1(P(Y)) \vee \ldots \vee N_S(P(Y))) \,\&\, (N_{S+1}(P(Y)) \vee \ldots \vee N_M(P(Y)))$. By theorem 4. 2, this contradicts our assumption of $Q(\text{A } 8)$. Since $N_D((X)P(X)) \,\&\, N_U(P(Y))$ denotes any term of our logical sum, our theorem will follow for A 8 by appropriate uses of the deduction theorem.

If we consider A 9, it is necessary to show that $Q(\text{A } 9) \supset \supset (N_1(\text{A } 9) \vee \ldots \vee N_S(\text{A } 9))$ is provable in the two-valued predicate calculus. As in the case of A 8, we assume $Q(\text{A } 9) \,\&\, \sim (N_1(\text{A } 9) \vee \vee \ldots \vee N_S(\text{A } 9))$ and deduce a contradiction. By theorem 4. 2, our assumption gives $N_{S+1}(\text{A } 9) \vee \ldots \vee N_M(\text{A } 9)$. Since A 9 is of

the form $R \supset S$ and $\supset (r, s)$ satisfies standard conditions, we can thus infer a logical sum of terms of the form $N_D((X)(P \supset Q(X)))$ & & $N_U(P \supset (X)Q(X))$ where D is a designated value and U is an undesignated value. Let $N_D((X)(P \supset Q(X)))$ & $N_U(P \supset (X)Q(X))$ denote any one of the terms of this logical sum. From such a term, one can infer a logical sum of terms of the form $N_{D*}(P)$ & & $N_{U*}((X)Q(X))$ where D^* is a designed value and U^* is an undesignated value. From this latter kind of term we get $N_{S+1}((X)Q(X)) \vee \vee \ldots \vee N_M((X)Q(X))$. But since $\supset (p, q)$ satisfies standard conditions, one can get $((N_1(P) \vee \ldots \vee N_S(P))$ & $(N_1(P \supset Q(X)) \vee \ldots \vee \vee N_S(P \supset Q(X))) \supset (N_1(Q(X) \vee \ldots \vee N_S(Q(X)))$ and so $((N_1(P) \vee \ldots \vee \vee N_S(P))$ & $(x)(N_1(P \supset Q(X)) \vee \ldots \vee N_S(P \supset Q(X))) \supset (x)(N_1(Q(X)) \vee \vee \ldots \vee N_S(Q(X)))$ since there is no free X in P. We thus have $((N_1(P) \vee \ldots \vee N_S(P))$ & $(N_1((X)(P \supset Q(X))) \vee \ldots \vee N_S((X)(P \supset \supset Q(X)))) \supset (N_1((X)Q(X)) \vee \ldots \vee N_S((X)Q(X)))$ since $N_r((X)P)$ satisfies standard conditions. Using this result with $N_D((X)(P \supset \supset Q(X)))$ it is easy to get $(N_1(P) \vee \ldots \vee N_S(P)) \supset N_1((X)Q(X)) \vee \ldots \vee \vee N_S((X)Q(X))$. Likewise, using this with $N_{D*}(P)$ one can get $N_1((X)Q(X)) \vee \ldots \vee N_S((X)Q(X))$. But this contradicts $N_{S+1}((X) Q(X)) \vee \ldots \vee N_M((X)Q(X))$ and our theorem will follow for A 9 by appropriate uses of the deduction theorem.

If we consider A 10, then as in the case of A 8 and A 9 we shall assume $Q(\text{A } 10)$ & $\sim (N_1(\text{A } 10) \vee \ldots \vee N_S(\text{A } 10))$ and deduce a contradiction. Again, from theorem 4. 2 we can get $N_{S+1}(\text{A } 10) \vee \vee \ldots \vee N_M(\text{A } 10)$. Hence, one can infer a logical sum of terms of the following form:

(1) $N_D(\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$ & $N_U(J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$, where D denotes a designated value and U denotes an undesignated value. This follows from the definition of partial normal forms for statements built up using $F_i(P_1, \ldots, P_{a_i})$ and the fact that $\supset (p, q)$ satisfies standard conditions.

Since $j_r(p)$ satisfies standard conditions, then from any one of the terms (1) we can infer a logical sum of terms of the following form:

(2) $N_t(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ where $t \neq r$. This follows from the definition of partial normal forms for statements which are built up using $F_i(P_1, \ldots, P_{a_i})$.

Now assume any one of the expressions $N_D(\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$ from (1). Note that $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$ is a W as defined in lemma 5. 1. 3 and that we have all of the assumptions of lemma 5. 1. 3 satisfied in the present case. Hence, using lemma 5. 1. 3 we can get,

(3) $(\sum_1^S N_i(\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))) \equiv (\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))^*$, where $(\mathbf{n}_r(\prod_i(X_1, \ldots X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))^*$ is just $N_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots P_{\gamma_i}))$. But since we have assumed $N_D(\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$, one can get

$$\sum_1^S N_i(\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$$

and so by (3) we have,

(4) $N_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$.

With suitable uses of the deduction theorem with (2) and (4) we can contradict Q(A 10). Hence, our theorem follows for A 10.

Case 2. Since $\supset (p, q)$ satisfies standard conditions, it is easy to see that condition (2) of the definition of plausible functions is satisfied.

Case 3. Consider condition (3) of the definition of plausible functions, and assume that P is acceptable by our truth-value stipulation. Then, $Q(P) \supset (N_1(P) \vee \ldots \vee N_S(P))$ is provable in the two-valued predicate calculus. If we now assume $Q((X)P)$ it is clear that we have $N_1(P) \vee \ldots \vee N_S(P)$, since $Q(P)$ and $Q((X)\,P)$ are identical. Since $N_r((X)P)$ satisfies standard conditions, we can thus get $N_1((X)P) \vee \ldots \vee N_S((X)P)$. Hence, by the deduction theorem of the two-valued predicate calculus we have $Q((X)P) \supset$ $\supset (N_1((X)P) \vee \ldots \vee N_S((X)P))$, so if P is acceptable by our truth-value stipulation, then so is $(X)P$, and condition (3) of the definition of plausible functions is satisfied. With this result established our theorem follows.

In view of lemma 5. 1. 2, we can now say that if the functions $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ are plausible, then our axiomatic stipulation is consistent. For example, a least statement is not acceptable according to our axiomatic stipulation. If it were and P is such a statement, then by lemma 5. 1. 2 we would have $Q(P) \supset (N_1(P) \vee \ldots \vee N_S(P))$ provable in the two-valued predicate calculus. But since P is a least statement, $N_1(P), \ldots, N_S(P)$ is the

same as $p_1, \ldots, p_S$ respectively. Hence, $Q(P) \supset (p_1 \vee \ldots \vee p_S)$ is not always "true", which contradicts the consistency of the ordinary two-valued predicate calculus, and the consistency of our axiomatic stipulation follows. Also, if the functions $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ satisfy standard conditions, then it will not be difficult for the reader to show that our axiomatic stipulation has the usual properties which are related to such concepts as that of $\sim$-consistency and $\sim$-decidability.

Later we will construct a large class of many-valued predicate calculi which are based on particular choices of basic statement functions and basic quantifiers. We will then make use of our general theorem 5. 1 in order to establish the plausibility of our particular predicate calculi. However, before turning to this kind of problem for particular many-valued logics, we will first consider the general question of the deductive completeness of our present axiomatic stipulation.

Because of A 10, the expressions $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_\gamma))$ play an essential part in our axiomatic stipulation for many-valued predicate calculi. As we have already remarked, $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ is patterned after $N_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$, where we assume that the latter partial normal form is built up out of $p_{1,k}, \ldots, p_{\gamma_i,k}$ using the two-valued operators $\supset$, $\sim$, and (x). Similarly, $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ is built up out of $J_k(P_1), \ldots, J_k(P_{\gamma_i})$ using the many-valued operators $\supset$, $\sim$, and (X). This is sufficient to indicate that the formal patterns of $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ and $N_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots P_{\gamma_i}))$ are the same, but for our present purposes we will need more than this formal similarity. In effect, we require that the truth-value properties of $\mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ shall be patterned after the analogy of the truth-value properties of $N_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$. This is achieved by choosing our many-valued $\supset$, $\sim$, (X), and $J_k(\ )$ in such a manner that their corresponding functions $\supset (p, q)$, $\sim (p)$, $N_r((X)P)$, and $j_k(p)$ satisfy standard conditions. In the proof of deductive completeness which follows, it will be assumed that such a choice has been made for our many-valued operators $\supset$, $\sim$, (X), and $J_k(\ )$. The results of the next chapter will indicate that this assumption is not as restrictive as one might at first imagine.

Let F_M denote a statement which is built up using the many-valued operators $\sim$ and $\supset$ in the same manner that a two-valued statement F_2 is built up using the two-valued operators $\sim$ and $\supset$. Since $\sim(p)$ and $\supset(p, q)$ satisfy standard conditions, a truth-table proof for F_M may be obtained from one for F_2 by replacing "true" and "false" in the given truth-table proof for F_2 by "designated" and "undesignated" respectively. For example, in two-valued logic one could give a truth-table proof of $P \supset ((\sim Q) \supset P)$ by showing that it is impossible for $P \supset ((\sim Q) \supset P)$ to be "false" and, hence, the given statement is provable by the deductive completeness of the two-valued propositional calculus. To see that the statement can not be "false", one could assume that P is "true" and $(\sim Q) \supset P$ is "false", and derive the contradictory result that P is "false". If we now replace "true" and "false" in the above by "designated" and "undesignated" respectively, our argument would show that since $\sim(p)$ and $\supset(p, q)$ satisfy standard conditions, the many-valued statement $P \supset ((\sim Q) \supset P)$ can not be "undesignated". Hence, since $\supset (p, q)$ and $j_k(p)$ satisfy standard conditions, our many-valued statement $P \supset ((\sim Q) \supset P)$ is provable by theorem 3. 4, i.e., $\vdash P \supset ((\sim Q) \supset P)$. In what follows, when use is made of this kind of argument to prove $\vdash F_M$, we will merely say that the proof follows by truth tables and theorem 3. 4.

Lemma 5. 2. 1. $\vdash (Q \supset (R \supset S)) \supset ((Q \supset R) \supset (Q \supset S))$.

Proof. Since $\supset (p, q)$ satisfies standard conditions, the lemma follows by truth tables and theorem 3. 4.

Theorem 5. 2. If $P_1, \ldots, P_n \vdash Q$ and no variable that is free in $P_1, \ldots, P_n$ is ever quantified by means of R 2, then we have $P_1, \ldots, P_{n-1} \vdash P_n \supset Q$. That is, we have a deduction theorem.

Proof. Since we have A 1, lemma 3. 1. 2, lemma 5. 2. 1, and A 9, the proof is strictly analogous to the proof of the deduction theorem in the two-valued case. [39]

Lemma 5. 3. 1. $\vdash (\sim P \supset \sim Q) \supset (Q \supset P)$.

Proof. Since $\supset (p, q)$ and $\sim (p)$ satisfy standard conditions, the lemma follows by truth tables and theorem 3. 4.

Theorem 5. 3. Let F_2 denote a statement of the two-valued predicate calculus which is built up out of statements $P_1, \ldots, P_n$

[39] See Church 1944, p. 45.

using the two-valued operators $\supset$, $\sim$, and (x). Let F_M denote a many-valued statement which results from F_2 by taking $P_1, \ldots, P_n$ as many-valued statements, and $\supset$, $\sim$, (x) as the many-valued operators $\supset$, $\sim$, and (X) respectively. Then, if F_2 is provable in the ordinary two-valued predicate calculus, then we have $\vdash F_M$.

Proof. There will be no loss of generality if Church's [40] F^1 is chosen as an axiomatic stipulation for the ordinary two-valued predicate calculus. However, if the axioms for F^1 are interpreted according to the assumptions of our theorem, then we obtain lemma 5. 2. 1, A 1, lemma 5. 3. 1, A 9, and A 8 respectively. Also, the rules of inference for F^1 will correspond exactly to our R 1 and R 2. Hence, any proof of F_2 in F^1 can be duplicated step by step to obtain a proof of $\vdash F_M$ and our theorem follows.

Note that if F_2 is provable in the two-valued predicate calculus under the assumptions $A_1, \ldots, A_a$, then the proof of theorem 5. 3 could be used to show that we can get a proof of $A_1, \ldots, A_a \vdash F_M$, where the assumptions $A_1, \ldots, A_a$ are now many-valued. In what follows, when the proof of theorem 5. 3 is thus used to prove $A_1, \ldots, A_a \vdash F_M$, we will say that the proof is by theorem 5. 3. If we now define many-valued &, $\vee$, and the logical sum $\sum$ in terms of our many-valued $\sim$ and $\supset$ in the same way that the ordinary &, $\vee$, and $\sum$ are defined in terms of the two-valued $\sim$ and $\supset$, we can use the many-valued operators &, $\vee$, and $\sum$ in the lemmas and theorems that follow. Henceforth, we will assume that the many-valued &, $\vee$, and $\sum$ have been defined in the manner specified.

Lemma 5. 4. 1. $P, Q \vdash P \,\&\, Q$.

Proof. This follows by theorem 5. 3.

Lemma 5. 4. 2. $P \vee Q,\ P \supset R,\ Q \supset S \vdash R \vee S$.

Proof. This follows by theorem 5. 3.

Lemma 5. 4. 3. $(P \,\&\, \sim Q) \supset (R \,\&\, \sim R) \vdash P \supset Q$.

Proof. This follows by theorem 5. 3.

Lemma 5. 4. 4. If $k_1 \neq k_2$, then $\vdash \sim (J_{k_1}(R) \,\&\, J_{k_2}(R))$.

Proof. Since $\supset(p, q)$, $\sim(p)$, and $j_k(p)$ satisfy standard conditions, this follows by truth tables and theorem 3. 4.

Lemma 5. 4. 5. If $k_1 \neq k_2$, then $\vdash (X_1, \ldots, X_n) \sim (J_{k_1}(R) \,\&\, J_{k_2}(R))$.

[40] Op. cit., p. 40. We assume that [IV] has been corrected.

Proof. Use lemma 5. 4. 4 and R 2.

Lemma 5. 4. 6. $\vdash \sum_{i=1}^{M} J_i(P)$.

Proof. Since $\supset(p, q)$ and $j_k(p)$ satisfy standard conditions, this follows by truth tables and theorem 3. 4.

Lemma 5. 4. 7. $\vdash (X_1, \ldots, X_n) \sum_{i=1}^{M} J_i(P)$.

Proof. Use lemma 5. 4. 6 and R 2.

Lemma 5. 4. 8. If A_j^M denotes the many-valued product of all factors of the form $(X_1, \ldots, X_n) \sim (J_{k_1}(P) \;\&\; J_{k_2}(P))$ for which $k_1 \neq k_2$ (k_1 and $k_2 = 1, \ldots, M$), and B_j^M denotes the many-valued sum $(X_1, \ldots, X_n) \sum_{i=1}^{M} J_i(P)$, then we have $\vdash A_j^M \;\&\; B_j^M$.

Proof. This follows by lemma 5. 4. 5, lemma 5. 4. 7, and lemma 5. 4. 1.

Lemma 5. 4. 9. $\vdash (Q \;\&\; P) \supset P$.

Proof. Since $\supset(p, q)$ and $\sim(p)$ satisfy standard conditions, use theorem 3. 4 and truth tables.

Lemma 5. 4. 10. For each i we have

$$\vdash \sum_{r=1}^{M} \boldsymbol{n}_r(\textstyle\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})).$$

Proof. By H 13 we have $(A_j \;\&\; B_j) \supset (C \;\&\; D)$ provable in the two-valued predicate calculus. Let A_j^M, B_j^M, C^M, and D^M denote the many-valed statements which result from replacing the $p_{j,k}$'s of H 13 by $J_k(P_j)$'s, and replacing the two-valued operators $\sim$, $\mathbf{v}$, and (x) used in constructing A_j, B_j, etc. by our many-valued $\sim$, $\supset$, and (X) respectively. Using theorem 5. 3, we then have $\vdash (A_j^M \;\&\; B_j^M) \supset (C^M \;\&\; D^M)$. But, A_j^M and B_j^M are the same as in lemma 5. 4. 8, so we can get $\vdash C^M \;\&\; D^M$. Since C^M is the many-valued logical product of all factors of the form $\sim(\boldsymbol{n}_{r_1}(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \;\&\; \boldsymbol{n}_{r_2}(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})))$ for which $r_1 \neq r_2$, and D^M is the many-valued sum $\sum_{r=1}^{M} \boldsymbol{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$, our lemma follows by lemma 5. 4. 9.

Lemma 5. 4. 11. If $n \geqslant k \geqslant 1$, then

$$(\sim P_k) \;\&\; \sum_{i=1}^{n} P_i \vdash \sum_{i=1}^{k-1} P_i \;\mathbf{v} \sum_{i=k+1}^{n} P_i.$$

Proof. Use theorem 5. 3.

Lemma 5. 4. 12. If $n \geqslant k \geqslant 1$, then

$$J_k(P) \;\&\; (\sum_{i=1}^{k-1} J_i(P) \;\mathbf{v} \sum_{i=k+1}^{M} J_i(P)) \vdash Q \;\&\; \sim Q.$$

Proof. Use theorem 5. 3.

Theorem 5. 4. For $1 \leqslant i \leqslant c$ and $1 \leqslant r \leqslant M$, $\vdash J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \supset \mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$.

Proof. Assume $J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \ \& \sim \mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$. By lemma 5. 4. 10 and 5. 4. 11, we can get

$$\vdash \sum_{j=1}^{r-1} \mathbf{n}_j(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \vee \sum_{j=r+1}^{M} \mathbf{n}_j(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})).$$

Hence, with repeated uses of lemma 5. 4. 2 and A 10 we can get

$$\vdash \sum_{j=1}^{r-1} J_j(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \vee \sum_{j=r+1}^{M} J_j(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})).$$

But by lemma 5. 4. 9 we have, $\vdash J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$. Hence, using lemmas 5. 4. 1 and 5. 4. 12 we can get $\vdash Q \ \& \sim Q$. If we now use theorem 5. 2 and lemma 5. 4. 3 our theorem follows.

Let us define a many-valued $\equiv$ as follows:

$$P \equiv Q =_{\mathrm{df}} (P \supset Q) \ \& \ (Q \supset P).$$

Theorem 5. 5. For $1 \leqslant i \leqslant c$ and $1 \leqslant r \leqslant M$ we have $\vdash J_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})) \equiv \mathbf{n}_r(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$.

Proof. Use lemma 5. 4. 1 with A 10 and theorem 5. 4.

Lemma 5. 6. 1. $\sim P \supset (Q \ \& \sim Q) \vdash P$.

Proof. Use theorem 5. 3.

Lemma 5. 6. 2. $P \supset (Q \ \& \sim Q) \vdash \sim P$.

Proof. Use theorem 5. 3.

Lemma 5. 6. 3. $\sim P_1 \ \& \ \ldots \ \& \sim P_n \vdash \sim \sum_{i=1}^{n} P_i$.

Proof. Use theorem 5. 3.

Define "a consistent set" of many-valued statements as follows:

A set of statements $\mathbf{s}$ is consistent if it is not the case that $\mathbf{s} \vdash Q \ \& \sim Q$. Otherwise $\mathbf{s}$ is inconsistent.

Define "a satisfiable set" of many-valued statements as follows:

A set of statements $\mathbf{s}$ is satisfiable if and only if there is a universe of individuals and a set of truth values such that each statement of $\mathbf{s}$ takes a designated truth value when the statements of $\mathbf{s}$ are

assigned truth functions of the individuals of the given universe and truth values according to the following plan:

(1) If P is in $\mathbf{s}$ and contains no free variables, then each occurrence of P in $\mathbf{s}$ is assigned the same designated truth value from our given set of truth values.

(2) If P is in $\mathbf{s}$ and there are free occurrences of $X_1, \ldots, X_n$ in P, then each occurrence of P in $\mathbf{s}$ is assigned the same function of individuals $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ which is so evaluated over the given universe of individuals that $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ takes a designated value.

Theorem 5. 6. If $\mathbf{s}$ is a consistent set, then $\mathbf{s}$ is satisfiable.

Proof. First introduce a set of distinct symbols $\mathbf{a}_{ij}(i$ and $j = 0, 1, 2, \ldots)$ which differ from any symbols that already occur in our many-valued predicate calculus. The set of $\mathbf{a}_{ij}$'s will constitute our given universe of individuals. Let S_0 denote the predicate calculus which is defined by our axiom schemes A 1 to A 10 and rules R 1 and R 2. Let S_1 denote the calculus obtained from S_0 by adding the constants $\mathbf{a}_{0j}$ to the list of individual variables for S_0 and otherwise leaving S_0 unaltered. In general, let S_{i+1} denote the calculus which is obtained from S_i by adding the constants $\mathbf{a}_{ij}$ to the list of individual variables for S_i and otherwise leaving S_i unaltered. Now let S_ω denote the calculus which results from taking all the S_i's together. The use which is here made of the "constants" $\mathbf{a}_{ij}$ is governed by the following restrictions. If P is a statement with free occurrences of X, and we replace these free occurrences of X by occurrences of $\mathbf{a}_{ij}$, the resulting formula is a statement. However, one never replaces bound occurrences of a variable by $\mathbf{a}_{ij}$. The axioms are the same except that A 8 is generalized so that Q can come from P by replacing free occurrences of X by $\mathbf{a}_{ij}$. In R 2 one never permits the prefixing of $(\mathbf{a}_{ij})$. Finally, let a distinct natural number be assigned to each statement of S_ω and arrange the statements of S_ω in the natural order of their assigned numbers [41].

We will now construct a "maximal consistent set" of the statements of S_0 which contain no free variables. First select a consistent set $\mathbf{s}$ of statements of S_0 which contain no free variables. For example, one could select the set $\{\text{A } 1\}^*$ where $\{\text{A } 1\}^*$ denotes the

[41] This could be done by using Gödel numbers. See Gödel 1931.

set of the instances of A 1 which contain no free variables. If this were not a consistent set, we would have $\Delta \vdash Q \mathbin{\&} \sim Q$ where Δ is a finite subset of $\{\text{A } 1\}^*$ and, hence, $\vdash Q \mathbin{\&} \sim Q$. But this contradicts the $\sim$-consistency of our present stipulation. We next examine one after another the statements of S_0 which contain no free variables, and accept or reject them according as they are or are not consistent with the statements of our chosen $\mathfrak{s}$ together with the statements previously accepted and added to $\mathfrak{s}$. The set of all the statements of $\mathfrak{s}$ containing no free variables which are accepted according to this procedure constitutes our maximal consistent set of statements from S_0 which contain no free variables, and will be denoted as $\mathfrak{G}_0$.

In order to construct a maximal consistent set $\mathfrak{G}_1$ of the statements of S_1 which contain no free variables, we select the first statement of $\mathfrak{G}_0$ in its natural order which is of the form $\sim (X) \sim P$. Let P^* denote the statement which results when each free occurrence of X in P is replaced by $\mathfrak{a}_{00}$. If P^* is added to $\mathfrak{G}_0$, then the resulting set is consistent. If it were not, we would have $\mathfrak{G}_0, P^* \vdash Q \mathbin{\&} \sim Q$, and by theorem 5. 2 and lemma 5. 6. 2 we could get $\mathfrak{G}_0 \vdash \sim P^*$. Since $\mathfrak{a}_{00}$ differs from each individual variable of S_0 and the statements of $\mathfrak{G}_0$ contain no free variables, we could duplicate the proof of $\mathfrak{G}_0 \vdash \sim P^*$ to get a proof of $\mathfrak{G}_0 \vdash \sim P$ if $\mathfrak{a}_{00}$ is replaced by X. Hence, by R 2 we have $\mathfrak{G}_0 \vdash (X) \sim P$, and $\mathfrak{G}_0$ is not consistent. In the same manner, we continue to choose one after another in their natural order the statements of $\mathfrak{G}_0$ which are of the form $\sim (X) \sim P$, and continue to enlarge the set $\mathfrak{G}_0$ by adding formulas P^* which result from replacing each free occurrence of X in P by an $\mathfrak{a}_{0j}$ such that a new $\mathfrak{a}_{0j}$ is selected for each choice of a statement of the form $\sim (X) \sim P$. After all such P^*'s have been added to $\mathfrak{G}_0$, the resulting set is enlarged to a maximal consistent set $\mathfrak{G}_1$ of the statements of S_1 which contain no free variables by applying the same kind of method which was used in constructing the set $\mathfrak{G}_0$.

In general, the method which is used to construct $\mathfrak{G}_1$ from $\mathfrak{G}_0$ can be applied in order to construct a maximal consistent set $\mathfrak{G}_{i+1}$ from a maximal consistent set $\mathfrak{G}_i$ by using the constants $\mathfrak{a}_{ij}$. For each i, we thus obtain a maximal consistent set $\mathfrak{G}_i$ of statements from S_i which contain no free variables. If $\mathfrak{G}_\omega$ denotes the union

of all the sets $\mathbf{G}_i$, then $\mathbf{G}_\omega$ is a maximal consistent set of the statements of S_ω which contain no free variables.

If $\mathbf{G}$ denotes any one of our maximal consistent sets of statements, then $\mathbf{G}$ has the following properties:

(1) If S is the calculus whose statements are used in constructing $\mathbf{G}$ and P is a statement of S which contains no free variables, then $\mathbf{G} \vdash \sim P$ if and only if P is not in $\mathbf{G}$. To see this, suppose $\mathbf{G} \vdash \sim P$ and P is in $\mathbf{G}$. We would then get $\mathbf{G} \vdash P \;\&\; \sim P$. Now, suppose that P is not in $\mathbf{G}$. This means that P has been rejected from $\mathbf{G}$ since it together with some subset $\varDelta$ of statements of $\mathbf{G}$ led to contradiction, i.e., $\varDelta, P \vdash Q \;\&\; \sim Q$. Hence, by theorem 5. 2 and lemma 5. 6. 2 we could get $\varDelta \vdash \sim P$. Since $\varDelta$ is a subset of $\mathbf{G}$, we have $\mathbf{G} \vdash \sim P$.

(2) If P is as in (1) above, then exactly one of $J_k(P)$ $(1 \leqslant k \leqslant M)$ is in $\mathbf{G}$. By lemma 5. 4. 4, it is clear that two distinct $J_k(P)$'s can not be in $\mathbf{G}$. Also, if each $J_k(P)$ is not in $\mathbf{G}$, then by (1) above we can get $\mathbf{G} \vdash \sim J_1(P), \ldots, \mathbf{G} \vdash \sim J_M(P)$. Hence, by lemma 5. 4. 1 we can get $\mathbf{G} \vdash \sim J_1(P) \;\&\; \ldots \;\&\; \sim J_M(P)$. But using this with lemma 5. 6. 3 and lemma 5. 4. 6 we can get $\mathbf{G} \vdash Q \;\&\; \sim Q$.

We will now assign truth values to the statements of S_ω according to the following rules:

(I) If P is a statement of S_ω which contains no free variables, then we shall assign to P the truth value k for which $J_k(P)$ is in $\mathbf{G}_\omega$. By (2) of the properties of maximal consistent sets, we know that there is such a k.

(II) If P is a statement of S_ω which contains the free variables $X_1, \ldots, X_n$, then assign to it the function of individuals $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ where the $\mathbf{a}_i$'s $(i = 1, \ldots, n)$ are variables over the domain of the $\mathbf{a}_{ij}$'s, and where $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is so defined that the truth value of $P(\mathbf{a}_{i_1 j_1}, \ldots, \mathbf{a}_{i_n j_n})$ is the k such that $J_k(P(\mathbf{a}_{i_1 j_1}, \ldots, \mathbf{a}_{i_n j_n}))$ is in $\mathbf{G}_\omega$. By (2) of the properties of maximal consistent sets we know that there is such a k, and we know that there is such a function $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ by our method of constructing $\mathbf{G}_\omega$.

We wish to show that by assigning truth values to the statements of S_ω according to our rules (I) and (II), our maximal consistent set $\mathbf{G}_\omega$ is satisfied. To this end let P be any statement of S_ω and let v denote the truth value which is assigned to P by our rules (I) and (II). Now consider the following points:

(1) If P is any statement of S_ω, then P is in $\mathbf{G}_\omega$ if and only if v is designated. For if v is designated, then $J_v(P)$ is in $\mathbf{G}_\omega$ by rule (I), and by A 6 we know that P is in $\mathbf{G}_\omega$. Also, if P is in $\mathbf{G}_\omega$ and v is not designated, then we could get $\mathbf{G}_\omega \vdash J_v(P) \mathbin{\&} P$. But since $\supset(p, q)$, $j_k(p)$, and $\sim(p)$ satisfy standard conditions, by theorem 3. 4 and truth tables we can get $\vdash (J_v(P) \mathbin{\&} P) \supset (Q \mathbin{\&} \sim Q)$, since v is not designated. Hence, we have the contradictory result that $\mathbf{G}_\omega \vdash Q \mathbin{\&} \sim Q$.

(2) If P has the structure $F_i(P_1, \ldots, P_{a_i})$, then let $v_1, \ldots, v_{a_i}$ denote the truth values which our rules (I) and (II) assign to $P_1, \ldots, P_{a_i}$ respectively. We then know that $J_u(P_k)$ is in $\mathbf{G}_\omega$ where $u = v_k$ and $k = 1, \ldots, a_i$. Also $J_v(P)$ is in $\mathbf{G}_\omega$. Hence, by A 7 we know that $v = f_i(v_1, \ldots, v_{a_i})$, for otherwise $\mathbf{G}_\omega$ is not consistent.

(3) If P has the structure $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$, then by theorem 5. 5 our rules (I) and (II) assign the truth value v to P if and only if $\mathbf{n}_v(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ is in $\mathbf{G}_\omega$. But by (1) above, P is assigned the value v if and only if $\mathbf{n}_v(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ takes a designated value. Since $\supset(p, q)$, $j_k(p)$, and $\sim(p)$ satisfy standard conditions, this means that P takes the value v if and only if $N_v(\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i}))$ is "true" over the given universe of individuals which is the domain of our $\mathbf{a}_{ij}$'s.

From (1), (2), and (3) above we know that each statement of $\mathbf{G}_\omega$ is assigned a designated truth value in a manner required by the conditions which define "satisfiable". Hence, the set $\mathbf{G}_\omega$ is satisfiable, and since $\mathbf{s}$ is a subset of $\mathbf{G}_\omega$ we know that $\mathbf{s}$ is satisfiable in our domain of $\mathbf{a}_{ij}$'s. Thus, our theorem 5. 6 is proved.

Lemma 5. 7. 1. If not-$\vdash P$, then the set $\{\sim P\}$ is consistent.

Proof. If $\{\sim P\}$ is not consistent, then $\sim P \vdash Q \mathbin{\&} \sim Q$. Hence, by theorem 5. 3 and lemma 5. 6. 1 we have $\vdash P$.

Lemma 5. 7. 2. If P^* is the closure [42] of P, then if not-$\vdash P$ then not-$\vdash P^*$.

Proof. If $\vdash P^*$ then by A 8 we can get $\vdash P$.

Lemma 5. 7. 3. If P^* is as in lemma 5. 7. 2 and the set $\{\sim P^*\}$ is satisfiable, then P is not acceptable by our truth-value stipulation.

[42] If $X_1, \ldots, X_n$ are the free variables which occur in P, then P^* is $(X_1, \ldots, X_n)P$.

Proof. If P is acceptable by our truth-value stipulation, then $Q(P) \supset (N_1(P) \vee \ldots \vee N_S(P))$ is provable in the two-valued predicate calculus. But, $Q((X)P) \supset Q(P)$ is also provable, and so $Q((X)P) \supset (x)(N_1(P) \vee \ldots \vee N_S(P))$ is provable in the two-valued predicate calculus. Since our $N_r((X)P)$ satisfies standard conditions, we can get $Q((X)P) \supset (N_1((X)P) \vee \ldots \vee N_S((X)P))$ provable, and, hence, P^* is acceptable by our truth-value stipulation. Since $\sim (p)$ satisfies standard conditions, it follows that $\sim P^*$ is not satisfiable. Hence, we have our lemma.

Theorem 5. 7. If P is acceptable by our truth-value stipulation, then $\vdash P$. That is our axiomatic stipulation is deductively complete.

Proof. If not-$\vdash P$, then by lemmas 5. 7. 2 and 5. 7. 1 we know that the set $\{\sim P^*\}$ is consistent. Hence, by theorem 5. 6 we know that the set $\{\sim P^*\}$ is satisfiable. Thus, our theorem follows by lemma 5. 7. 3.

Our proof of deductive completeness is a generalization of Henkin's proof of the deductive completeness of the ordinary two-valued predicate calculus [43]. Also, one can generalize Gödel's proof of the deductive completeness of the two-valued predicate calculus [44] so that it will apply to many-valued predicate calculi [45]. However, since Henkin's proof is the more elegant, we will not attempt to generalize the Gödel proof in the present work.

Theorem 5. 8. If the functions $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ satisfy standard conditions, then our axiomatic stipulation is equivalent to our truth-value stipulation.

Proof. This is an immediate consequence of theorems 5. 1 and 5. 7.

If we continue to assume that $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ satisfy standard conditions, then as a corollary of our theorem 5. 6 we can prove the following:

Corollary. A set of many-valued statements which contain no free variables and which is satisfiable in some domain of individuals is satisfiable in a denumerable domain of individuals.

Proof. Let **F** be a set of many-valued statements which contain no free variables. If **F** is satisfiable in some domain of individuals,

[43] See Henkin 1949.

[44] See Gödel 1930.

[45] See Rosser and Turquette 1951.

then $\boldsymbol{F}$ is consistent. Otherwise, $\boldsymbol{F} \vdash Q \,\&\, \sim Q$ which means that $Q \,\&\, \sim Q$ is either in $\boldsymbol{F}$ or results from members of $\boldsymbol{F}$ by the use of our rules R 1 and R 2. In either case, $\boldsymbol{F}$ is not satisfiable in any domain since $\supset(p, q)$, $\sim(p)$, and $N_r((X)P)$ satisfy standard conditions. Hence, $\boldsymbol{F}$ must be a consistent set, and by our theorem 5. 6 we know that $\boldsymbol{F}$ is satisfiable in the denumerable domain of the $\boldsymbol{a}_{ij}$'s of theorem 5. 6.

With the proof of our corollary, we have a many-valued generalization of the ordinary two-valued Löwenheim–Skolem theorem[46]. This concludes our general treatment of axiomatic stipulations for many-valued predicate calculi. In the next chapter we will apply this general theory to some particular many-valued predicate calculi.

[46] Skolem 1929.

VI

QUANTIFICATION FOR PARTICULAR PREDICATE CALCULI

By applying the results of the last two chapters, it is possible to introduce quantifiers into special systems of many-valued logic. That is, for a given M and S it is possible to construct many-valued predicate calculi which are based on particular choices for the functions $F_i(P_1, \ldots, P_{\alpha_i})$ and $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. The chief purpose of the present chapter is to illustrate such an application of our general theory of quantification for many-valued logics.

In accordance with the prevailing theme of the present work, our illustrations of many-valued predicate calculi will be confined to systems which are deductively complete [47]. In fact, attention will be focused on systems which are such that their axiomatic stipulation is equivalent to a given truth-value stipulation. However, we shall not be so restrictive regarding functional completeness. We have already observed that for $M > 2$ statement calculi which are based on the Łukasiewicz–Tarski operators C and N (F_1 and F_2) are not functionally complete. In spite of this fact, our first illustrations will be that of introducing quantifiers into many-valued logics which are based on C and N.

First illustration. For this example we will choose F_1 and F_2 as in chapter II and consider the case of a logical system with any choice for M but in which $S = 1$. Recalling that $f_1(p_1, p_2) = \max(1, p_2 - p_1 + 1)$ and $f_2(p_1) = M - p_1 + 1$, we see that our hypotheses H 1 to H 7 are satisfied. Next, it will be necessary to satisfy our hypotheses H 8 to H 13, and to this end we must choose our functions $\prod_i(X_1, \ldots, X_{\beta_i}, P_1, \ldots, P_{\gamma_i})$. In the present case, we shall take $c = \beta_i = \gamma_i = 1$ in H 9, and, hence, choose the single function $\prod_1(X_1, P_1)$ as basic. If we use individual variables and classify them as free and bound in strict analogy to the two-

[47] For an example of a many-valued predicate calculus which is not deductively complete see Barcan 1946.

valued case, then we will have satisfied our hypotheses H 8 to H 12. Thus, if we can satisfy H 13, then in the present case all our hypotheses H 1 to H 13 will be satisfied.

In satisfying H 13, we will be guided by the heuristic principle involved in thinking of $\prod(X, P(X))$ as the logical product of every $P(X)$ as X ranges over all possible values. Hence, $\prod(X, P(X))$ may be thought of as the greatest truth value of $P(X)$ for any X. Using this interpretation of $\prod(X, P(X))$, we could describe its truth-value properties as follows:

(1) $\prod(X, P(X))$ takes the truth-value j $(1 \leqslant j \leqslant M)$ if and only if there is an X for which $P(X)$ takes the truth-value j and for every X, $P(X)$ takes a truth-value t such that $t \leqslant j$.

A moment's reflection will indicate that (1) provides us with the following set of partial normal forms for $\prod(X, P(X))$:

$N_1(\prod(X, P(X)))$ is $(\exists x)N_1(P(X))$ & $(x)N_1(P(X))$
$N_2(\prod(X, P(X)))$ is $(\exists x)N_2(P(X))$ & $(x)(N_1(P(X)) \vee N_2(P(X)))$
$\vdots$
$N_M(\prod(X, P(X)))$ is $(\exists x)N_M(P(X))$ & $(x)(N_1(P(X)) \vee \ldots \vee N_M(P(X)))$.

If we take the above as our definition of the rth partial normal form of $\prod(X, P(X))$, then $N_r(\prod(X, P(X)))$ can be written in terms of the two-valued operators $\sim$, (x), and $\supset$. We will show that such partial normal forms enable us to satisfy H 13 for the present illustration.

To this end, let $S_n(X)$ denote the two-valued expression $N_1(P(X)) \vee \ldots \vee N_n(P(X))$. We can then write our $N_r(\prod(X, P(X)))$ as follows:

$N_1(\prod(X, P(X)))$ is $(\exists x)N_1(P(X))$ & $(x)S_1(X)$
$N_2(\prod(X, P(X)))$ is $(\exists x)N_2(P(X))$ & $(x)S_2(X)$
$\vdots$
$N_M(\prod(X, P(X)))$ is $(\exists x)N_M(P(X))$ & $(x)S_M(X)$

Lemma 6. 1. 1. In the ordinary two-valued predicate calculus one can prove $((x)S_n(X)) \supset (N_1(\prod(X, P(X))) \vee \ldots \vee N_n(\prod(X,P(X))))$.

Proof. Use induction on n.

(α) If $n = 1$, then we must prove $(x)S_1(X) \supset N_1(\prod(X, P(X)))$. But this follows by the definition of $S_1(X)$ and $N_1(\prod(X, P(X)))$.

(β) Assume our theorem for $n = k$ and prove it for $n = k + 1$. By the definition of $S_n(X)$, we have $(x)S_{k+1}(X) \supset (x)(S_k(X) \vee N_{k+1}(P(X)))$ provable in the two-valued predicate calculus. Hence, we can get $(x)S_{k+1}(X) \supset ((\exists x)N_{k+1}(P(X)) \vee (x)S_k(X))$, and by the two-valued propositional calculus we have $(x)S_{k+1}(X) \supset ((x)S_k(X) \vee ((\exists x)N_{k+1}(P(X)) \,\&\, (x)S_{k+1}(X)))$. But $((\exists x)N_{k+1}(P(X))) \,\&\, (x)S_{k+1}(X)$ is just $N_{k+1}(\prod(X, P(X)))$, so we have

(1) $(x)S_{k+1}(X) \supset ((x)S_k(X) \vee N_{k+1}(\prod(X, P(X))))$. By assumption ($\beta$), we also have

(2) $(x)S_k(X) \supset (N_1(\prod(X, P(X))) \vee \ldots \vee N_k(\prod(X, P(X)))$. If we now use (1) and (2) with the two-valued propositional calculus our theorem will follow for $n = k + 1$.

Lemma 6. 1. 2. If $t < n$, then in the ordinary two-valued predicate calculus one can prove $(x) \sim (N_1(P(X)) \,\&\, N_n(P(X))) \,\&\, (x) \sim (N_2(P(X)) \,\&\, N_n(P(X))) \,\&\, \ldots \,\&\, (x) \sim (N_t(P(X)) \,\&\, N_n(P(X))) \supset \sim (N_t(\prod(X, P(X)) \,\&\, N_n(\prod(X, P(X))))$.

Proof. Assume $t < n$. We can then get $N_t(\prod(X, P(X))) \,\&\, N_n(\prod(X, P(X))) \supset (x)S_t(X) \,\&\, (\exists x)N_n(P(X))$ provable in the two-valued predicate calculus by the definition of $S_n(X)$ and the definition of the rth partial normal form of $\prod(X, P(X))$. Hence, we have $N_t(\prod(X, P(X))) \,\&\, N_n(\prod(X, P(X))) \supset (\exists x)(S_t(X) \,\&\, N_n(P(X)))$ provable. So we can get $N_t(\prod(X, P(X))) \,\&\, N_n(\prod(X, P(X))) \supset (\exists x)((N_1(P(X)) \,\&\, N_n(P(X))) \vee (N_2(P(X)) \,\&\, N_n(P(X))) \vee \ldots \vee (N_t(P(X)) \,\&\, N_n(P(X))))$. Hence, we have $N_t(\prod(X, P(X))) \,\&\, N_n(\prod(X, P(X))) \supset \sim (x)(\sim (N_1(P(X)) \,\&\, N_n(P(X))) \,\&\, \sim (N_2(P(X)) \,\&\, N_n(P(X))) \,\&\, \ldots \,\&\, \sim (N_t(P(X)) \,\&\, N_n(P(X))))$. By contraposition, we can get $(x)(\sim (N_1(P(X)) \,\&\, N_n(P(X))) \,\&\, \ldots \,\&\, \sim (N_t(P(X)) \,\&\, N_n(P(X)))) \supset \sim (N_t(\prod(X, P(X))) \,\&\, N_n(\prod(X, P(X))))$. From this it is not difficult to deduce our theorem.

Theorem 6. 1. If A_1, B_1, C, and D are as defined in H 13, then $(A_1 \,\&\, B_1) \supset (C \,\&\, D)$ is provable in the two-valued predicate calculus. That is, since in the present case we have $\gamma_i = 1$, it follows that H 13 is satisfied.

Proof. By definition, A_1 is the logical product of all factors of the form $(x) \sim (N_{r_1}(P(X)) \,\&\, N_{r_2}(P(X)))$ where $r_1 \neq r_2$ (r_1 and

$r_2 = 1, \ldots, M$), and B_1 is $(x)(N_1(P(X)) \vee \ldots \vee N_M(P(X)))$. Also, by definition C is the logical product of all factors of the form $\sim (N_{r_1^*}(\prod(X, P(X))) \mathbin{\&} N_{r_2^*}(\prod(X, P(X))))$ where $r_1^* \neq r_2^*$ (r_1^* and $r_2^* = 1, \ldots, M$), and D is $N_1(\prod(X, P(X))) \vee \ldots \vee N_M(\prod(X, P(X)))$. Hence, by lemma 6. 1. 2 we can get $A_1 \supset C$ provable in the two-valued predicate calculus. Also, by lemma 6. 1. 1 we have $B_1 \supset D$ provable in the two-valued predicate calculus. Thus, it is not difficult to deduce our theorem, and it follows that we have H 13 satisfied for our present illustration.

With the proof of theorem 6. 1, we have satisfied our hypotheses H 1 to H 13. It will now be shown that with our choice for M, S, and the set of basic functions, we have some functions $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ which satisfy standard conditions. To this end, a many-valued universal quantifier (X) will be defined as follows:

$$(X)P =_{\text{df}} \prod(X, P).$$

Lemma 6. 2. 1. If (X) is defined as above, then $N_r((X)P)$ satisfies standard conditions.

Proof. By definition, it is easy to see that we have $(x)N_1(P(X)) \equiv N_1((X)P(X))$ provable in the two-valued predicate calculus. Hence, our lemma follows since $S = 1$.

In what follows since $P \supset Q$ interpreted as $\overline{P} \vee Q$ is of special significance, we shall introduce an abbreviation for $\overline{P} \vee Q$ as indicated by the following definition:

$$P \; I \; Q =_{\text{df}} \overline{P} \vee Q.$$

Theorem 6. 2. For the present case, there are functions $\supset (p, q)$, $j_k(p)$, $\sim (p)$, and $N_r((X)P)$ which satisfy standard conditions.

Proof. If we take I, $J_k(\ )$, and $\overline{}$ which are defined in terms of F_1 and F_2 in chapter II as $\supset$, $J_k(\ )$, and $\sim$ respectively, then $\supset (p, q)$, $j_k(p)$, and $\sim (p)$ satisfy standard conditions. Recall that $P \vee Q$ is $F_1(F_1(P, Q), Q)$, and $\overline{P}$ is $J_{S+1}(P) \vee \ldots \vee J_M(P)$. Note also that $J_k(\ )$ has the desired truth-value properties by theorems 2. 3 and 2. 4. Hence, our theorem follows by lemma 6. 2. 1.

Theorem 6. 3. There is an axiomatic stipulation which is equivalent to the truth-value stipulation defined by our present choice for M, S, and the set of basic functions.

Proof. Since we have satisfied our hypotheses H 1 to H 13, use the axiom schemes A 1 to A 10 and rules R 1 and R 2 of chapter V with the $\supset$, $J_k(\ )$, $\sim$, and (X) which appear in the axiom schemes and rules interpreted as in the proof of theorem 6. 2. Our present theorem will then follow by theorem 6. 2 and theorem 5. 8.

Actually, in the present case an axiomatic stipulation can be given which is more elegant than the one indicated in the proof of theorem 6. 3. In effect, the more elegant axiomatic stipulation which we have in mind results by omitting A 10 from the axiom schemes of chapter V and assigning an interpretation to $\supset$ which is different from the one used in the proof of theorem 6. 3. In fact, $\supset$ is taken to be C (that is, F_1). Since A 10 is omitted, there is no need for $\sim$ in the axiom schemes, but $J_k(\ )$ and (X) are used and are chosen to be the same as in the proof of theorem 6. 3. In order to see just how this comes about, we will define the axioms and rules for our more elegant axiomatic stipulation and show that it is equivalent to our given truth-value stipulation.

To this end, our axiomatic stipulation will be defined by the following rules:

(1) Our axiom schemes are A 1 to A 9 with $\supset$, $J_k(\ )$, and (X) interpreted as F_1, $J_k(\ )$ as defined in chapter II in terms of F_1 and F_2, and (X) as defined above in terms of $\prod(X, P)$ respectively.

(2) Our rules of inference are R 1 and R 2 with $\supset$ and (X) interpreted as in (1) above.

We will show first that our axiomatic stipulation is plausible relative to our given truth-value stipulation. Later we shall show that it is also deductively complete relative to our truth-value stipulation. From these two results it will then follow that our axiomatic and truth-value stipulations are equivalent.

Theorem 6. 4. The axiomatic stipulation defined by (1) and (2) above is plausible relative to the truth-value stipulation defined by the choice for M, S, and the set of basic functions for the present illustration.

Proof. Suppose that W is acceptable according to the axiomatic stipulation defined by (1) and (2). We then have the following cases:

Case 1. If W is an instance of any one of the axiom schemes A 1 to A 7, then by theorem 3. 5 we know that W always takes the truth value 1. Hence, by the same kind of argument that was

used in case 1 of the proof of theorem 5. 1 we know that our pressent theorem follows.

Case 2. If W is an instance of A 8 or A 9, then we will sketch the proof for A 8 and leave it to the reader to check A 9. Hence, we will assume $Q(\text{A } 8)$ & $\sim N_1(\text{A } 8)$ and deduce a contradiction. From our assumption we can get $N_2(\text{A } 8) \vee \ldots \vee N_M(\text{A } 8)$. Thus, we obtain a logical sum of terms of the form $N_a((X)P(X))\&N_b(P(Y))$ where $b > a$. Assume any such term. We can then get

$$(x)(N_1(P(X)) \vee \ldots \vee N_a(P(X)))$$

and, hence, have

$$(N_1(P(Y)) \vee \ldots \vee N_a(P(Y))) \ \& \ N_b(P(Y)).$$

Since $b > a$, from this latter expression we can contradict $Q(\text{A } 8)$. Thus, our theorem will follow by appropriate uses of the deduction theorem.

Case 3. If W results from the use of R 1 or R 2, then since $\supset$ is F_1 and $N_r((X)P)$ satisfies standard conditions, it is not difficult to verify the fact that our theorem follows in this case.

To show that the axiomatic stipulation defined by (1) and (2) is deductively complete relative to our given truth value stipulation, it is necessary to establish a rather large number of preliminary results. Our strategy will be to show the acceptability by our present axiomatic stipulation of A 1 to A 10, R 1, and R 2 when the $\supset$, $J_k(\ )$, $\sim$, and (X) which appear in the axiom schemes and rules are taken as I, $J_k(\ )$ as defined in terms of F_1 and F_2 in chapter II, $\overline{}$, and the (X) as defined above in terms of $\prod(X, P)$ respectively. We then can use theorem 5. 7 to infer the deductive completeness of our axiomatic stipulation defined by (1) and (2) relative to our given truth-value stipulation. This procedure will require the use of two interpretations of $\supset$. In order to distinguish these, we will use $\supset$ to denote only F_1 and will write I when this form of implication is intended. Hence, $Q \supset (P \supset Q)$ is one of the axiom schemes of our axiomatic stipulation and is the same as $F_1(Q,\ F_1(P,\ Q))$, while we hope to prove $Q\ I\ (P\ I\ Q)$ using our axiomatic stipulation.

In view of theorem 3. 5, truth tables may be used to establish a large class of statements which are acceptable by our axiomatic

stipulation. If a proof is said to be established by truth tables in our present illustration, then it is to be understood that reference is being made to the use of theorem 3. 5. For example, the acceptability of $Q\ I\ (P\ I\ Q)$ by our present axiomatic stipulation can be proved by truth tables. That is, A 1 is acceptable when $\supset$ is taken as I. In like fashion, one can show that A 1 to A 7 are acceptable by the axiomatic stipulation defined by (1) and (2) when $\supset$ is taken as I and $J_k(\ )$ has the same interpretation as in (1) of the definition of our axiomatic stipulation. The statement of this fact is the purpose of the next theorem.

Theorem 6. 5. If A 1 to A 7 are interpreted as we have just indicated, then they are acceptable by the axiomatic stipulation which is defined by (1) and (2).

Proof. The theorem follows by truth tables.

In view of theorem 6. 5, if we can now prove A 8, A 9, A 10, R 1, and R 2 with $\supset$ and $J_k(\ )$ interpreted as in theorem 6. 5, and $(X)P$ and $\sim$ interpreted as $\prod(X, P)$ and $\overline{}$ respectively, then the deductive completeness of the axiomatic stipulation defined by (1) and (2) will follow by theorem 5. 7. The remainder of our first illustration will be devoted to the task of establishing such a result. For this purpose, we shall assume as in chapter II that $P \vee Q$ is $F_1(F_1(P, Q), Q)$ and $P \,\&\, Q$ is $F_2(F_2(P) \vee F_2(Q))$. Also, we assume that the definition of our yields sign has been altered to fit the case of the axiomatic stipulation which has been defined by (1) and (2).

Define a many-valued $\equiv$ as follows:

$$P \equiv Q =_{\mathrm{df}} (P \supset Q) \,\&\, (Q \supset P)$$

Lemma 6. 6. 1. $\vdash (P \,\&\, Q) \supset P$ and $\vdash (Q \,\&\, P) \supset P$.

Proof. Use truth tables.

Lemma 6. 6. 2. $\vdash (P \equiv Q) \supset (F_2(P) \equiv F_2(Q))$.

Proof. Use truth tables.

Lemma 6. 6. 3. $\vdash P \supset (Q \supset (P \,\&\, Q))$.

Proof. Use truth tables.

Lemma 6. 6. 4. $P \equiv Q,\ R \equiv S \vdash (P \supset R) \equiv (Q \supset S)$.

Proof. By lemma 6. 6. 1 we have,

(1) $P \equiv Q,\ R \equiv S \vdash R \supset S$.

Using (1) and lemma 3. 1. 1 one can get,

(2) $P \equiv Q,\ R \equiv S \vdash (Q \supset R) \supset (Q \supset S)$.

But we also have,

(3) $P \equiv Q,\ R \equiv S \vdash Q \supset P$.

Hence, (3) with A 3 gives,

(4) $P \equiv Q,\ R \equiv S \vdash (P \supset R) \supset (Q \supset R)$.

By (2), (4), and A 3 we have,

(5) $P \equiv Q,\ R \equiv S \vdash (P \supset R) \supset (Q \supset S)$.

Clearly, by a similar argument one can get,

(6) $P \equiv Q,\ R \equiv S \vdash (Q \supset S) \supset (P \supset R)$.

Now use (5), (6), and lemma 6. 6. 3.

Lemma 6. 6. 5. $\vdash (X)(P \supset Q) \supset ((X)P \supset (X)Q)$.

Proof. Using A 3 we can get,

(1) $\vdash ((X)P \supset P) \supset ((P \supset Q) \supset ((X)P \supset Q))$.

Hence, (1), A 8, and R 1 will give,

(2) $\vdash (P \supset Q) \supset (((X)P) \supset Q)$.

But by A 8 we also have,

(3) $\vdash (X)(P \supset Q) \supset (P \supset Q)$.

Hence, (2), (3), and A 3 give,

(4) $\vdash (X)(P \supset Q) \supset (((X)P) \supset Q)$.

If we apply. R 2 to (4) and use A 9 we can get,

(5) $\vdash (X)(P \supset Q) \supset (X)(((X)P) \supset Q)$.

If we use (5), A 9, and A 3 our lemma follows.

We can now prove a substitution theorem for $\equiv$. This is the purpose of the next theorem.

Theorem 6. 6. Let $P_1, \ldots, P_n$, R, and Q be statements of unspecified structure, some of which may contain individual variables. Let W be built up out of P_j's $(1 \leqslant j \leqslant n)$ and R by means of $\supset$ (that is, F_1), F_2, and (X) (that is, $\prod(X, P)$). Let W^* denote the result of replacing some or none of the R's in W by Q's. We then have,

$$R \equiv Q \vdash W \equiv W^*.$$

Proof. Use induction on the structure of W.

(α) If W is a single symbol, then it is either a P_j or R. If it is a P_j, then it is easy to get $R \equiv Q \vdash P_j \equiv P_j$ and our theorem follows. If W is R and R is replaced by Q, then we can get $R \equiv Q \vdash R \equiv Q$ and the theorem follows. If W is R and R is not replaced by Q, then we can get $R \equiv Q \vdash R \equiv R$ and our theorem follows.

(β) Assume the theorem for W with k or less symbols and let W have $k + 1$ symbols. From the structure of W, we have the following cases:

Case 1. If W has the structure $F_2(W_1)$, then W^* is $F_2(W_1^*)$. But by assumption (β) we have,

(1) $R \equiv Q \vdash W_1 \equiv W_1^*$.

Using (1) and lemma 6. 6. 2 we have $R \equiv Q \vdash F_2(W_1) \equiv F_2(W_1^*)$ and our theorem follows.

Case 2. If W has the structure $W_1 \supset W_2$, then W^* is $W_1^* \supset W_2^*$. But by assumption (β) we have,

(1) $R \equiv Q \vdash W_1 \equiv W_1^*$.

(2) $R \equiv Q \vdash W_2 \equiv W_2^*$.

Using (1), (2), and lemma 6. 6. 4 we can get,

$$R \equiv Q \vdash (W_1 \supset W_2) \equiv (W_1^* \supset W_2^*).$$

Hence, our theorem follows.

Case 3. If W has the structure $(X)W_1$, then W^* is $(X)W_1^*$. But by assumption (β) we have,

(1) $R \equiv Q \vdash W_1 \equiv W_1^*$.

Using (1) and lemma 6. 6. 1 we can get,

(2) $R \equiv Q \vdash W_1 \supset W_1^*$.

Hence, applying R 2 to (2) and using lemma 6. 6. 5 will give,

(3) $R \equiv Q \vdash (X)W_1 \supset (X)W_1^*$.

By a similar argument one can get,

(4) $R \equiv Q \vdash (X)W_1^* \supset (X)W_1$.

Hence, from (3) and (4) our theorem will follow by lemma 6. 6. 3.

Lemma 6. 7. 1. If there are no free X's in P, then

$$\vdash (X)(P \supset Q) \equiv (P \supset (X)Q).$$

Proof. By A 8 and lemma 3. 1. 1 we have,

(1) $\vdash (P \supset (X)Q) \supset (P \supset Q)$.

Applying R 2 to (1) and using A 9 we can get,

(2) $\vdash (P \supset (X)Q) \supset (X)(P \supset Q)$.

Hence, our lemma follows from (2), A 9, and lemma 6. 6. 3.

Lemma 6. 7. 2. If there are no free X's in P, then

$$\vdash (X)(P \vee Q) \supset (P \vee (X)Q).$$

Proof. Recalling that $P \vee Q$ is just $(P \supset Q) \supset Q$, we have by lemma 6. 6. 5,

(1) $\vdash (X)(P \vee Q) \supset ((X)(P \supset Q) \supset (X)Q)$.

If we now use (1) with lemma 6. 7. 1 and theorem 6. 6 our lemma follows.

Lemma 6. 7. 3. $\vdash P \supset (Q \vee P)$.

Proof. Use truth tables.

Lemma 6. 7. 4. $\vdash P \supset (P \vee Q)$.

Proof. Use truth tables.

Lemma 6. 7. 5. $\vdash (P \vee P) \supset P$.

Proof. Use truth tables.

Lemma 6. 7. 6. $\vdash (P \supset Q) \supset ((R \supset S) \supset ((P \vee R) \supset (Q \vee S)))$.

Proof. Use truth tables.

Lemma 6. 7. 7. If there are no free X's in P, then

$$\vdash (X)(P \vee Q) \equiv (P \vee (X)Q).$$

Proof. By A 8, lemma 6. 7. 3, and A 3 we have,

(1) $\vdash (X)Q \supset (P \vee Q)$.

Applying R 2 to (1) and using A 9 we can get,

(2) $\vdash (X)Q \supset (X)(P \vee Q)$.

By lemma 6. 7. 4, we also have $\vdash P \supset (P \vee Q)$ where there are no free X's in P. Hence, using R 2 and A 9 we can get,

(3) $\vdash P \supset (X)(P \vee Q)$.

But by using (2) and (3) with lemmas 6. 7. 6 and 6. 7. 5 we can get,

(4) $\vdash (P \vee (X)Q) \supset (X)(P \vee Q)$.

If we now use (4) and lemma 6. 7. 2 with lemma 6. 6. 3 our lemma follows.

Define $F_1^k(P, Q)$ as follows:

(α) $F_1^0(P, Q) =_{\text{df}} Q$.

(β) $F_1^{k+1}(P, Q) =_{\text{df}} P \supset F_1^k(P, Q)$.

Lemma 6. 7. 8. If $f_1^k(p, q)$ is the truth function associated with $F_1^k(P, Q)$, then $f_1^k(p, q) = \max(1, q - k(p - 1))$.

Proof. Use induction on k.

(α) If $k = 0$, then $F_1^0(P, Q)$ is Q and $f_1^0(p, q) = q = \max(1, q-k(p - 1))$. Hence, our lemma follows.

(β) Assume the lemma for k and prove it for $k + 1$. By definition, $F_1^{k+1}(P, Q)$ is $P \supset F_1^k(P, Q)$. Hence, $f_1^{k+1}(p, q) = \max(1, f_1^k(p, q) - - p + 1)$. By assumption ($\beta$) it then follows that $f_1^{k+1}(p, q) = \max(1, \max(1, q - k(p - 1)) - p + 1) = \max(1, q - k(p - 1) - (p - 1)) = = \max(1, q - (k + 1)(p - 1))$. Hence, our lemma follows.

Lemma 6. 7. 9. $\vdash (P\ I\ Q) \equiv F_1^{M-1}(P, Q)$.

Proof. Use truth-tables and consider the following cases:

Case 1. If $p = 1$, then $I(p, q) = q$ and by lemma 6. 7. 8, $f_1^{M-1}(p, q) = \max(1, q) = q$.

Case 2. If $p \neq 1$, then $I(p, q) = 1$ and by lemma 6. 7. 8, $f_1^{M-1}(p, q) = \max(1, q - (M - 1)(p - 1)) = 1$.

Lemma 6. 7. 10. $P, P\ I\ Q \vdash Q$.

Proof. This follows by R 1 and lemma 6. 7. 9.

Lemma 6. 7. 11. $\vdash P\ I\ P$.

Proof. Use truth tables.

Lemma 6. 7. 12. $\vdash P \supset (Q\ I\ P)$.

Proof. Use truth tables.

Lemma 6. 7. 13. $\vdash (P\ I\ Q)\ I\ ((P\ I\ (Q \supset R))\ I\ (P\ I\ R))$.

Proof. Use truth tables.

Lemma 6. 7. 14. If there are no free X's in P, then

$$\vdash (X)(P\ I\ Q) \equiv (P\ I\ (X)Q).$$

Proof. If we put $\overline{P}$ for P in lemma 6. 7. 7, our present lemma will follow by the definition of I.

We are now in a position to prove a deduction theorem in terms of I. This is the purpose of the next theorem.

Theorem 6. 7. If we have $P_1, \ldots, P_n, Q \vdash R$, where the proof is such that no variable which occurs free in Q is ever quantified by means of R 2, then we can get a proof of $P_1, \ldots, P_n \vdash Q\ I\ R$.

Proof. Let $\mathfrak{s}$ denote any step in the proof of $P_1, \ldots, P_n, Q \vdash R$. We will show how one can obtain a corresponding step of the form

$Q \, I \, \mathbf{s}$ in the proof of $P_1, \ldots, P_n \vdash Q \, I \, R$. Consider the following cases:

Case 1. If $\mathbf{s}$ is either an axiom or a $P_j(1 \leqslant j \leqslant n)$, then use lemma 6. 7. 12 and R 1.

Case 2. If $\mathbf{s}$ is Q, then use lemma 6. 7. 11.

Case 3. If $\mathbf{s}$ results from two earlier steps $\mathbf{s}_1$ and $\mathbf{s}_2$ by the use of R 1, then suppose that $\mathbf{s}_1$ is $\mathbf{s}_2 \supset \mathbf{s}$. Hence, if we have $Q \, I \, \mathbf{s}_1$ and $Q \, I \, \mathbf{s}_2$, one can use lemmas 6. 7. 10 and 6. 7. 13 to obtain $Q \, I \, \mathbf{s}$.

Case 4. If $\mathbf{s}$ results from an earlier step $\mathbf{s}_1$, then $\mathbf{s}$ is $(X)\mathbf{s}_1$. Hence, if we have $Q \, I \, \mathbf{s}_1$, one can use R 2 and lemma 6. 7. 14 to obtain $Q \, I \, \mathbf{s}$.

Since all cases have been considered, it is easy to see that an argument by induction will establish our theorem.

Lemma 6. 8. 1. If P and Q are as in A 8, then $\vdash ((X)P) \, I \, Q$.

Proof. By A 8 we have $(X)P \vdash Q$. If we now use theorem 6. 7 our lemma follows.

Lemma 6. 8. 2. If P and Q are as in A 9, then

$$\vdash (X)(P \, I \, Q) \, I \, (P \, I \, (X)Q).$$

Proof. By lemma 6. 7. 14, we can get $(X)(P \, I \, Q) \vdash P \, I \, (X)Q$. If we now use theorem 6. 7 our lemma will follow.

Note that lemmas 6. 7. 10, 6. 8. 1, and 6. 8. 2 give us an R 1, A 8, and A 9 respectively which will allow us to use theorem 5. 7 to infer deductive completeness after we obtain an A 10 and R 2 of proper form. Since R 2 is already in proper form, all that remains to be shown is that we have an A 10 of the desired type. The rest of the first illustration will be devoted to the task of proving that we have such an A 10.

Define a quantifier $(\sum X)$ as follows:

$$(\textstyle\sum X)P =_{\mathrm{df}} F_2((X)F_2(P)).$$

Lemma 6. 8. 3. $\vdash P \equiv F_2(F_2(P))$.

Proof. Use truth tables.

Lemma 6. 8. 4. $\vdash (P \supset Q) \equiv (F_2(Q) \supset F_2(P))$.

Proof. Use truth tables.

Lemma 6. 8. 5. If P and Q are as in A 8, then $\vdash Q \supset (\sum X)P$.

Proof. In A 8 put $F_2(P)$ for P and then use lemmas 6. 8. 4 and 6. 8. 3 with theorem 6. 6.

Lemma 6. 8. 6. If there are no free X's in P, then

$$\vdash (X)(Q \supset P) \equiv ((\sum X)Q \supset P).$$

Proof. In lemma 6. 7.1 put $F_2(P)$ and $F_2(Q)$ for P and Q respectively. Now use lemmas 6. 8. 3 and 6. 8. 4 with theorem 6. 6.

Lemma 6. 8. 7. $\vdash (P \vee Q) \equiv (Q \vee P)$.

Proof. Use truth tables.

Lemma 6. 8. 8. If there are no free X's in P, then

$$\vdash (((\sum X)(Q \supset P)) \supset P) \equiv ((((X)Q) \supset P) \supset P).$$

Proof. Since $P \vee Q$ is $(P \supset Q) \supset Q$, we can use lemmas 6. 7. 7 and 6. 8. 7 with theorem 6. 6 to get,

(1) $\vdash (X)((Q \supset P) \supset P) \equiv (((X)Q \supset P) \supset P)$. Now use (1) with lemma 6. 8. 6 and theorem 6. 6.

Lemma 6. 8. 9. $\vdash (P \supset Q)\, I\, ((Q \supset R)\, I\, (((Q \supset P) \equiv (R \supset P))\, I$ $(Q \equiv R)))$.

Proof. Use truth tables. Note that by the definition of I, if $\supset(p, q)$ or $\supset(q, r)$ or $\equiv(\supset(q, p), \supset(r, p)) \neq 1$, then the lemma follows. Hence, suppose that $\supset(p, q) = \supset(q, r) = \equiv(\supset(q, p), \supset(r, p)) = 1$. We then know that $p \geqslant q$ and $q \geqslant r$. Thus, $\supset(q, p) = p - q + 1$ and $\supset(r, p) = p - r + 1$. But $\equiv(p - q + 1, p - r + 1) = 1$ and so $p - q + 1 = p - r + 1$. Hence, $q = r$ and $\equiv(q, r) = 1$. With this our lemma follows.

Lemma 6. 8. 10. If there are no free X's in P, then

$$\vdash P \supset (\sum X)(Q \supset P).$$

Proof. By A 1 we have $\vdash P \supset (Q \supset P)$ and lemma 6. 8. 5 gives $\vdash (Q \supset P) \supset (\sum X)(Q \supset P)$. Now use A 3.

Lemma 6. 8. 11. If there are no free X's in P, then

$$\vdash (\sum X)(Q \supset P) \supset (((X)Q) \supset P).$$

Proof. Using A 8 and A 3 we can get $\vdash (Q \supset P) \supset (((X)Q) \supset P)$. Now use R 2 and lemma 6. 8. 6.

Lemma 6. 8. 12. If there are no free X's in P, then

$$\vdash (\sum X)(Q \supset P) \equiv (((X)Q) \supset P).$$

Proof. In lemma 6. 8. 9 put $(\sum X)(Q \supset P)$ for Q and $((X)Q) \supset P$ for R. Now use lemmas 6. 8. 10, 6. 8. 11, 6. 8. 8, and 6. 8. 9.

Lemma 6. 8. 13. If there are no free X's in P, then

$$\vdash (\sum X)(P \supset Q) \equiv (P \supset (\sum X)Q).$$

Proof. In lemma 6. 8. 12 put $F_2(Q)$ and $F_2(P)$ for Q and P respectively. Now use lemmas 6. 8. 4 and 6. 8. 3 with theorem 6. 6.

Lemma 6. 8. 14. If there are no free X's in P, then

$$\vdash (((\sum X)Q) \,\&\, P) \equiv (\sum X)(Q \,\&\, P).$$

Proof. In lemma 6. 7. 7 put $F_2(P)$ and $F_2(Q)$ for P and Q respectively. By lemma 6. 6. 2 we can then get,

(1) $\vdash F_2((X)(F_2(P) \vee F_2(Q))) \equiv F_2(F_2(P) \vee (X)F_2(Q))$.

Using (1) with lemma 6. 8. 3 and theorem 6. 6 one can obtain,

(2) $\vdash F_2((X)F_2(F_2(F_2(P) \vee F_2(Q)))) \equiv F_2(F_2(P) \vee$
$F_2(F_2((X)F_2(Q))))$.

Recalling the definitions of $(\sum X)$ and &, it is easy to infer our lemma.

We will now define a certain class of functions, and use $\theta(P)$ throughout the next six lemmas to denote an unspecified function of this class.

$\theta(P)$ is built up out of $P, Q_1, \ldots, Q_n$ by means of F_1 and F_2 alone. We assume that P and any of the Q's may contain (X)'s but no (X) is used in building up $\theta(P)$. Further, $\theta(P)$ contains exactly one occurrence of P. We will then classify $\theta(P)$ as "positive" or "negative" according to the following rules:

(A) If $\theta(P)$ is P, then it is positive.

(B) If $\theta(P)$ is positive, then $F_2(\theta(P))$ is negative.

(C) If $\theta(P)$ is negative, then $F_2(\theta(P))$ is positive.

(D) If $\theta(P)$ is positive and ψ does not contain P, then $\psi \supset \theta(P)$ is positive and $\theta(P) \supset \psi$ is negative.

(E) If $\theta(P)$ is negative and ψ does not contain P, then $\psi \supset \theta(P)$ is negative and $\theta(P) \supset \psi$ is positive.

Lemma 6. 8. 15. If $\theta(P)$ is a positive function of P and we replace any set of $Q_1, \ldots, Q_n$ by any set of R's, and replace P by an S' different from any of the Q's and R's, then the resulting function of S' is positive. A similar result holds for negative functions.

Proof. This is an immediate consequence of the definition of positive and negative functions $\theta(P)$.

Lemma 6. 8. 16. Let $\theta(P)$ be a positive or negative function of P. Assign a set of truth values to $Q_1, \ldots, Q_n$ and let the truth values of P vary. We then have the following:

(a) If $\theta(P)$ is positive and we let the truth values of P decrease, then the truth values of $\theta(P)$ will not increase.

(b) If $\theta(P)$ is negative and we let the truth values of P decrease, then the truth values of $\theta(P)$ will not decrease.

Proof. Use induction on the structure of $\theta(P)$.

(α) If $\theta(P)$ is P, then the lemma is obvious.

(β) Assume the lemma for $\theta(P)$ with less than k symbols and let $\theta(P)$ have k symbols. Consider the following cases:

Case 1. If $\theta(P)$ is of the form $F_2(\theta^*(P))$ and is positive, then $\theta^*(P)$ is negative. By assumption (β), the truth values of $\theta^*(P)$ will not decrease. Hence, the truth values of $\theta(P)$ will not increase. If $\theta(P)$ is negative, then $\theta^*(P)$ is positive. By assumption (β), the truth values of $\theta^*(P)$ will not increase. Hence, the truth values of $\theta(P)$ will not decrease.

Case 2. If $\theta(P)$ is of the form $\psi \supset \theta^*(P)$ and is positive, then $\theta^*(P)$ is positive. By assumption (β), the truth values of $\theta^*(P)$ will not increase. Hence, the truth values of $\theta(P)$ do not increase. If $\theta(P)$ is negative, $\theta^*(P)$ is negative. By assumption (β), the truth values of $\theta^*(P)$ will not decrease. Hence, the truth values of $\theta(P)$ will not decrease.

Case 3. If $\theta(P)$ is of the form $\theta^*(P) \supset \psi$, then by an argument similar to that used in case 2 it is not difficult to show that our lemma follows.

Lemma 6. 8. 17. Let $\theta(P)$ be a positive or negative function of P and assume that none of the Q's contain a free X. We then have the following:

(a) If $\theta(P)$ is positive, then $\vdash \theta((X)P^*) \equiv (X)\theta(P^*)$.

(b) If $\theta(P)$ is negative, then $\vdash \theta((X)P^*) \equiv (\sum X)\theta(P^*)$.

Proof. Use strong induction on the structure of $\theta(P)$.

(α) If $\theta(P)$ is P, then the lemma is obvious since we have $P \supset P$.

(β) Assume the lemma for less than k symbols and let $\theta(P)$ have k symbols. Consider the following cases:

Case 1. If $\theta(P)$ is of the form $F_2(\theta^*(P))$ and is positive, then $\theta^*(P)$ is negative. Also, we have,

(1) $\vdash \theta((X)P^*) \equiv F_2(\theta^*((X)P^*))$.

By assumption (β) we can get,

(2) $\vdash \theta^*((X)P^*) \equiv (\sum X)\theta^*(P^*)$.

Hence, (1), (2), and theorem 6. 6 will give,

(3) $\vdash \theta((X)P^*) \equiv F_2(F_2((X)F_2(\theta^*(P^*))))$.

If we now use (3), lemma 6. 8. 3, and theorem 6. 6 our lemma follows.

If $\theta(P)$ is negative, then $\theta^*(P)$ is positive.
Hence, in place of (3) above we have,

(3*) $\vdash \theta((X)P^*) \equiv F_2((X)\theta^*(P^*))$.

Using (3*) with lemma 6. 8. 3 and theorem 6. 6 we have,

(4*) $\vdash \theta((X)P^*) \equiv F_2((X)F_2(F_2(\theta^*(P^*))))$.

Hence, our lemma follows.

Case 2. If $\theta(P)$ is of the form $\psi \supset \theta^*(P)$ and is positive, then $\theta^*(P)$ is positive. Also, we have,

(1) $\vdash \theta((X)P^*) \equiv (\psi \supset \theta^*((X)P^*))$.

By (1), assumption (β), and theorem 6. 6 we can get,

(2) $\vdash \theta((X)P^*) \equiv (\psi \supset (X)\theta^*(P^*))$.

Since there are no free X's in ψ, by (2), lemma 6. 7. 1, and theorem 6. 6 our lemma follows.

If $\theta(P)$ is negative, then $\theta^*(P)$ is negative.
Hence, in place of (2) we can get,

(2*) $\vdash \theta((X)P^*) \equiv (\psi \supset (\sum X)\theta^*(P^*))$.

If we now use (2*) with lemma 6. 8. 13 and theorem 6. 6 our lemma follows.

Case 3. If $\theta(P)$ is of the form $\theta^*(P) \supset \psi$ and is positive, then $\theta^*(P)$ is negative. Also, we have,

(1) $\vdash \theta((X)P^*) \equiv (\theta^*((X)P^*) \supset \psi)$.

By (1), assumption (β), and theorem 6. 6 we can get,

(2) $\vdash \theta((X)P^*) \equiv (((\sum X)\theta^*(P^*)) \supset \psi)$.

Hence, by (2), lemma 6. 8. 6, and theorem 6. 6 our lemma follows.

If $\theta(P)$ is negative, then $\theta^*(P)$ is positive.
Hence, in place of (2) above we will get,

(2*) $\vdash \theta((X)P^*) \equiv (((X)\theta^*(P^*)) \supset \psi)$.

If we now use (2*) with lemma 6. 8. 12 and theorem 6. 6 our lemma follows.

Lemma 6. 8. 18. For each k $(1 \leqslant k \leqslant M)$, there is a function $J_k(P_1, \ldots, P_{a_k}, Q_1, \ldots, Q_{b_k})$ which has the following properties:
(1) $J_k(P, \ldots, P, P, \ldots, P)$ is just $J_k(P)$.
(2) $J_k(P_1, \ldots, P_{a_k}, Q_1, \ldots, Q_{b_k})$ is a positive function $\theta(P)$ of each P_i $(1 \leqslant i \leqslant a_k)$ and a negative function $\theta(P)$ of each Q_i $(1 \leqslant i \leqslant b_k)$.

Proof. In the definition of $J_k(P)$ in chapter II, replace the successive occurrences of P by $R_1, \ldots, R_{a_k+b_k}$, where the R's are all distinct. The resulting formula is built up out of R's by means of F_1 and F_2, and it contains each R exactly once. Hence, the resulting formula is a positive function $\theta(P)$ of certain of the R's and a negative function of the rest of the R's. Now, in this resulting formula replace the R's for which it is positive by P's and the R's for which it is negative by Q's, taking care not to replace any two R's by the same letter. By lemma 6. 8. 15, we thus have property (2) of our present lemma. It is also clear from our method of constructing our function, that if the P's and Q's are each replaced by P, then we have $J_k(P)$. Hence, we have property (1) and our lemma follows.

Lemma 6. 8. 19. If $X_1, \ldots, X_{a_k}, Y_1, \ldots, Y_{b_k}$ are distinct variables, then $\vdash J_k((X)P) \equiv (\sum Y_1) \ldots (\sum Y_{b_k})(X_1) \ldots (X_{a_k})J_k(P(X_1), \ldots, P(X_{a_k}), P(Y_1), \ldots, P(Y_{b_k}))$.

Proof. By lemmas 6. 8. 15 and 6. 8. 18 we note that $J_k(P(X_1), \ldots, P(X_{a_k}), P(Y_1), \ldots, P(Y_{b_k}))$ is a positive function of $P(X_{a_k})$. Hence, by lemma 6. 8. 17 we have $\vdash (X_{a_k})J_k(P(X_1), \ldots, P(X_{a_k}), P(Y_1), \ldots, P(Y_{b_k})) \equiv J_k(P(X_1), \ldots, (X_{a_k})P(X_{a_k}), P(Y_1), \ldots, P(Y_{b_k}))$. By such successive uses of lemmas 6. 8. 15, 6. 8. 18, and 6. 8. 17 it is not difficult to establish our lemma.

Lemma 6. 8. 20. If a finite sum $\sum$ and product $\prod$ is defined in the usual manner, then

$$\vdash (J_k(Q) \,\&\, (\prod_{j=1}^{a_k} \sum_{i=1}^{k} J_i(P_j))) \supset J_k(P_1, \ldots, P_{a_k}, Q, \ldots, Q).$$

Proof. Use truth values. Note that one can confine attention to those values of $P_1, \ldots, P_{a_k}, Q$ which make $J_k(Q) \,\&\, (\prod_{j=1}^{a_k} \sum_{i=1}^{k} J_i(P_j))$ take a truth value different from M. The only way that this is

possible is for Q to take the value k and the P's to take truth values which are not greater than k. Hence, let Q and the P's be assigned such a set of truth values. By lemma 6. 8. 18, if Q and the P's all take the truth value k then $J_k(P_1, \ldots, P_{a_k}, Q, \ldots, Q)$ takes the truth value 1. Now reduce the truth value of P_1 from k to whatever value P_1 does have according to our assigment, but simultaneously keeping the values of Q and the other P's fixed at k. Since $J_k(P_1, \ldots, P_{a_k}, Q, \ldots, Q)$ is a positive function of P_1, it follows by lemma 6. 8. 16 that its truth value remains fixed at 1. Now hold P_1 at the value it was reduced to, hold Q and $P_3, \ldots, P_{a_k}$ at the value k, and reduce the truth value of P_2 from k to whatever value P_2 does have according to our assignment. Again, by lemma 6. 8. 16 the truth value of $J_k(P_1, \ldots, P_{a_k}, Q, \ldots, Q)$ remains fixed at 1. By continuing this process, one can successively reduce the values of the P's from k to the correct value according to our assignment. Then by using lemma 6. 8. 16 one can show at each step that $J_k(P_1, \ldots, P_{a_k}, Q, \ldots, Q)$ takes a truth value which remains fixed at 1. Hence, our lemma follows.

Lemma 6. 8. 21. $\vdash ((P \supset Q_1) \& \ldots \& (P \supset Q_n)) \supset (P \supset (Q_1 \& \ldots \& Q_n))$.

Proof. Use truth tables.

Lemma 6. 8. 22. If none of $X_1, \ldots, X_{a_k}$ are free in Q, then $\vdash (J_k(Q) \& ((X) \sum_{i=1}^{k} J_i(P(X)))) \supset (X_1) \ldots (X_{a_k}) J_k(P(X_1), \ldots, P(X_{a_k}), Q, \ldots, Q)$.

Proof. Using A 8, lemmas 6. 6. 3 and 6. 8. 21 we can get,

(1) $\vdash (J_k(Q) \& ((X) \sum_{i=1}^{k} J_i(P(X)))) \supset (J_k(Q) \& \prod_{j=1}^{a_k} \sum_{i=1}^{k} J_i(P(X_j)))$.

If we now use (1) with lemma 6. 8. 20, our lemma will follow by successive uses of R 2 and A 9.

Lemma 6. 8. 23. $\vdash P(X, \ldots, X) \supset (\sum Y_1) \ldots (\sum Y_n) P(Y_1, \ldots, Y_n)$.

Proof. This follows by successive uses of lemma 6. 8. 5.

Lemma 6. 8. 24. $\vdash ((\sum X) J_k(P(X)) \& (X) \sum_{i=1}^{k} J_i(P(X))) \supset$
$\supset J_k((X)P(X))$.

Proof. By lemma 6. 8. 22 we have,

(1) $\vdash (J_k(P(X)) \& ((X) \sum_{i=1}^{k} J_i(P(X)))) \supset (X_1) \ldots (X_{a_k}) J_k(P(X_1), \ldots, P(X_{a_k}), P(X), \ldots, P(X))$.

Using (1) and lemma 6. 8. 23 we can get,

(2) $\vdash (J_k(P(X)) \& ((X) \sum_{i=1}^{k} J_i(P(X)))) \supset (\sum Y_1) \ldots (\sum Y_{b_k})(X_1) \ldots (X_{a_k}) J_k(P(X_1), \ldots, P(X_{a_k}), P(Y_1), \ldots, P(Y_{b_k}))$.

Now use (2) with lemma 6. 8. 19 and theorem 6. 6 to obtain

(3) $\vdash (J_k(P(X)) \mathbin{\&} ((X) \sum_{i=1}^{k} J_i(P(X)))) \supset J_k((X)P(X))$.

If we now apply R 2 to (3) and use lemmas 6. 8. 6 and 6. 8. 14 with theorem 6. 6 our lemma follows.

Lemma 6. 8. 25. $\vdash (P\ I\ Q)\ I\ (\overline{Q}\ I\ \overline{P})$.

Proof. Use truth tables.

Lemma 6. 8. 26. $\vdash \overline{\overline{P}}\ I\ P$.

Proof. Use truth tables.

Lemma 6. 8. 27. $\vdash (P\ I\ (Q\ I\ \overline{\overline{R}}))\ I\ (P\ I\ (Q\ I\ R))$.

Proof. Use truth tables.

Define a quantifier (EX) as follows:

$$(EX)P =_{\mathrm{df}} \overline{(X)\overline{P}}$$

Lemma 6. 8. 28. If there are no free X's in P, then

$$\vdash (X)(Q\ I\ P)\ I\ (((EX)Q)\ I\ P).$$

Proof. By A 8, lemmas 6. 8. 25, and 6. 7. 10 we have,

(1) $(X)(Q\ I\ P) \vdash \overline{P}\ I\ \overline{Q}$.

Applying R 2 to (1) and using lemma 6. 7. 14 we get,

(2) $(X)(Q\ I\ P) \vdash \overline{P}\ I\ (X)\overline{Q}$.

Using (2) and lemma 6. 8. 25 we have,

(3) $(X)(Q\ I\ P) \vdash ((EX)Q)\ I\ \overline{\overline{P}}$.

Now use (3) and theorem 6. 7 to get,

(4) $\vdash (X)(Q\ I\ P)\ I\ (((EX)Q)\ I\ \overline{\overline{P}})$.

If we now use (4) with lemma 6. 8. 27 our lemma follows.

Lemma 6. 8. 29. $\vdash ((EX)P)\ I\ ((\sum X)P)$.

Proof. By lemma 6. 8. 5 and theorem 6. 7 we have $\vdash P\ I\ (\sum X)P$. Now use R 2 and lemma 6. 8. 28.

Theorem 6. 8. $\vdash (((EX)J_k(P(X))) \mathbin{\&}$

$$(X) \sum_{i=1}^{k} J_i(P(X)))\ I\ J_k((X)P(X)).$$

Proof. Assume $((EX)J_k(P(X)) \mathbin{\&} (X) \sum_{i=1}^{k} J_i(P(X)))$. We can then get $(EX)J_k(P(X))$ and $(X) \sum_{i=1}^{k} J_i(P(X))$. Hence, by using lemma 6. 8. 29 we can get $((\sum X)J_k(P(X))) \mathbin{\&} (X) \sum_{i=1}^{k} J_i(P(X))$. If we now use lemma 6. 8. 24 and theorem 6. 7 our present theorem follows.

Theorem 6. 9. The axiomatic stipulation defined by (1) and (2) is deductively complete relative to the truth-value stipulation of the first illustration.

Proof. Note that by theorem 6. 8, we have $\vdash$ A 10 where A 10 has the desired interpretation. Hence, our theorem follows by using theorem 5. 7 with lemmas 6. 7. 10, 6. 8. 1, 6. 8. 2, and theorem 6. 5.

Theorem 6. 10. The axiomatic stipulation defined by (1) and (2) is equivalent to our given truth-value stipulation.

Proof. Use theorem 6. 9 and theorem 6. 4.

Second illustration. In this example, we shall first proceed exactly as in the first illustration except that now S is taken as greater than 1, and alterations in the argument of the first illustration are made only as they are required by this different choice for S. Hence, H 1 to H 13 are satisfied, and the first change resulting from our new choice for S is introduced in connection with the choice for the functions $\supset(p, q)$, $j_k(p)$, $\sim(p)$, and $N_r((X)P)$ which satisfy standard conditions. Now $\overline{P}$ is $\sum_{S+1}^{M} J_i(P)$ where $S > 1$ with corresponding changes involved in taking $\supset$ as I. However, no changes are made in the choices for $J_k(\;)$ and (X). Hence, we shall need a new lemma 6. 2. 1* in the place of lemma 6. 2. 1.

Lemma 6. 2. 1*. If (X) is defined as indicated and $S > 1$, then $N_r((X)P)$ satisfies standard conditions.

Proof. By lemma 6. 1. 1 we have,

$$(1) \quad (x)(N_1(P) \vee \ldots \vee N_S(P)) \supset (N_1((X)P) \vee \ldots \vee N_S((X)P)).$$

To get the converse of (1), assume $N_j((X)P)$ $(1 \leqslant j \leqslant S)$. We then can get $(x)(N_1(P) \vee \ldots \vee N_j(P))$ and so $N_1(P) \vee \ldots \vee N_j(P)$. Hence, we have $N_1(P) \vee \ldots \vee N_S(P)$ and $(x)(N_1(P) \vee \ldots \vee N_S(P))$. So by the deduction theorem, $N_j((X)P) \supset (x)(N_1(P) \vee \ldots \vee N_S(P))$. Since this result holds for every j with $1 \leqslant j \leqslant S$, it is not difficult to get the converse of (1), and our lemma follows.

With the proof of lemma 6. 2. 1*, it is clear that we can get a theorem 6. 2 and 6. 3 for our present illustration. All that is required is to take $J_k(\;)$ and (X) as before and use the new definitions of $\sim$ and $\supset$ as indicated above by the choice of $S > 1$.

Also, as in the first illustration, we can give an axiomatic stipulation which is more elegant that the one just indicated, and which is equivalent to the truth-value stipulation defined for our present illustration. To this end, first consider the axiomatic stipulation defined by the following rules:

(1) Our first seven axiom schemes are A 1 to A 7 with $J_k(\ )$ defined as in the first illustration, but now with $\supset$ interpreted as I for $S > 1$.

(2) Our next two axiom schemes are J_1(A 8) and J_1(A 9), where A 8 and A 9 are interpreted as in rule (1) of the more elegant axiomatic stipulation of the first illustration.

(3) Our last axiom scheme is as follows:

$$(L) \quad ((X)J_1(P))\ I\ J_1((X)P).$$

(4) Our rules R 1 and R 2 are the same as in rule (2) of the first illustration except that $\supset$ is now defined as I for $S > 1$.

Making use of theorem 3. 4 and the truth-value properties of $J_k(\ )$ and I, it will not be difficult for the reader to verify the fact that by proceeding as in the proof of theorem 6. 4, one can obtain a corresponding theorem 6. 4* for our present axiomatic and truth-value stipulations. Hence, we have plausibility for our second illustration.

In order to establish deductive completeness, the same strategy will be used as in the first illustration. Note that by (1) of the definition of our present axiomatic stipulation, we immediately have a theorem 6. 5* corresponding to 6. 5 of the first illustration. Since our present R 1 and R 2 are already in the desired form, our present problem of deductive completeness will be solved if one can show that proper forms of A 8, A 9, and A 10 are acceptable according to the axiomatic stipulation defined above by (1) to (4). To this end, we shall now prove some theorems, and in so doing theorem 3. 4 will be used in the same manner that theorem 3. 5 was used in the first illustration to establish results by truth tables. Unless otherwise specified, it is to be assumed that all definitions are as in the first illustration. Let $\vdash_1$ denote a yields sign as defined by rules (1) and (2) in our first illustration, and let $\vdash_2$ denote a yields sign as defined by (1) to (4) in our present illustration.

Lemma 6. 11. 1. $\vdash_2 P \supset (Q \supset (P \ \&\ Q))$.

Proof. Use truth tables.

Lemma 6. 11. 2. $\vdash_2 (J_1(P) \ \&\ J_1(F_1(P, Q))) \supset J_1(Q)$.

Proof. Use truth tables.

Theorem 6. 11. If $\vdash_1 P$, then $\vdash_2 J_1(P)$.

Proof. Let $\mathbf{s}_1, \ldots, \mathbf{s}_n$ denote a demonstration of $\vdash_1 P$. We will show how to get a demonstration $\mathbf{R}_1, \ldots, \mathbf{R}_m$ of $\vdash_2 J_1(P)$. Use induction on the steps in the demonstration of $\vdash_1 P$.

(α) If the demonstration of $\vdash_1 P$ consists of a single step, then P is $\mathbf{s}_1$ and $\mathbf{s}_1$ is an axiom as defined by (1) and (2) in the first illustration. Consider the following cases:

Case 1. If $\mathbf{s}_1$ is an instance of any one of the axiom schemes A 1 to A 7, then we know by the proof of theorem 6. 5 that $\mathbf{s}_1$ always takes the truth value 1. Hence, we can use truth tables to get $\vdash_2 J_1(P)$.

Case 2. If $\mathbf{s}_1$ is an instance of either A 8 or A 9, then we can use J_1(A 8) and J_1(A 9) to get $\vdash_2 J_1(P)$.

(β) Assume our theorem for all demonstrations of $\vdash_1 P$ with n or less steps and prove it for demonstrations with $n + 1$ steps. Let $\mathbf{s}_1, \ldots, \mathbf{s}_{n+1}$ be a demonstration of $\vdash_1 P$ with $n + 1$ steps. If P is an axiom use (α). If P results from the use of a rule, consider the following cases:

Case 1. If P results from the use of R 1, then there is a step $\mathbf{s}_i$ ($i < n + 1$) and a step $\mathbf{s}_j$ ($j < n + 1$) of our demonstration of $\vdash_1 P$ such that $\mathbf{s}_j$ is $F_1(\mathbf{s}_i, \mathbf{s}_{n+1})$ ($\mathbf{s}_{n+1}$ is P). But $\mathbf{s}_1, \ldots, \mathbf{s}_i$ and $\mathbf{s}_1, \ldots, \mathbf{s}_j$ are demonstrations of $\vdash_1 \mathbf{s}_i$ and $\vdash_1 \mathbf{s}_j$ respectively which have less than $n + 1$ steps. Hence, by assumption (β) we have $\vdash_2 J_1(\mathbf{s}_i)$ and $\vdash_2 J_1(\mathbf{s}_j)$. If we use this result with lemmas 6. 11. 1 and 6. 11. 2, then our theorem follows.

Case 2. If P results from the use of R 2, then there is a step $\mathbf{s}_i$ ($i < n + 1$) of the demonstration of $\vdash_1 P$ which is such that P is $(X)\mathbf{s}_i$. But $\mathbf{s}_1, \ldots, \mathbf{s}_i$ is a demonstration of $\vdash_1 \mathbf{s}_i$ with less than $n + 1$ steps. Hence, by assumption (β) we have $\vdash_2 J_1(\mathbf{s}_i)$. By R 2, we get $\vdash_2 (X)J_1(\mathbf{s}_i)$. Using this result with (L) and R 1 our theorem follows.

We can now prove a deduction theorem for our present interpretation of $\supset$. This is the purpose of our next theorem.

Theorem 6. 12. If $P_1, \ldots, P_n, Q \vdash_2 R$, where the use of R 2 is restricted as in theorem 6. 7, then $P_1, \ldots, P_n \vdash_2 Q \supset R$.

Proof. Using our axioms and truth tables, we can get,

(1) $\vdash_2 P \supset P$.

(2) $\vdash_2 P \supset (Q \supset P)$.

(3) $\vdash_2 (P \supset Q) \supset ((P \supset (Q \supset R)) \supset (P \supset R))$.

Hence, we can establish our theorem if we can prove,

(4) $\vdash_2 (X)(P \supset Q) \supset (P \supset (X)Q)$ where there are no free X's in P.

In order to prove (4), use theorem 6. 11 with lemma 6. 7. 2 to get,

(5) $\vdash_2 J_1(F_1((X)(P \vee Q),\ P \vee (X)Q))$.

If in (5) we replace P by $\overline{P}$ and use the definition of $\supset$, then we have,

(6) $\vdash_2 J_1(F_1((X)(P \supset Q),\ P \supset (X)Q))$.

But by truth tables one has,

(7) $\vdash_2 J_1(F_1(P, Q)) \supset (P \supset Q)$.

Hence, our theorem follows by (6) and (7).

Note that (4) in the proof of theorem 6. 12 gives us our desired form of A 9. The next theorem will give us our A 8.

Theorem 6.13. If P and Q are as in A8, then we have $\vdash_2 (X)P \supset Q$.

Proof. By truth tables we get,

(1) $\vdash_2 (P \ \&\ J_1(F_1(P, Q))) \supset Q$.

Also, by A 8 of our first illustration and theorem 6. 11 we have,

(2) $\vdash_2 J_1(F_1((X)P, Q))$.

Hence, (1) and (2) give,

(3) $(X)P \vdash_2 Q$.

Now use theorem 6. 12.

If we can establish $\vdash_2$ A 10, then our problem will be solved. To this end, define (EX) as in the first illustration but with $\overline{}$ given its altered definition for $S > 1$.

Theorem 6. 14. $\vdash_2$ A 10.

Proof. Since by theorem 6. 11 we have $\vdash_2 J_1$ (Lemma 6. 8. 24), one can establish our present theorem in much the same manner that theorem 6. 8 was proved in the first illustration. It will be necessary to insert $J_1(\ )$ at appropriate places, but we shall leave this instructive exercise to the reader.

Finally, note that in the present case a still more elegant axiomatic stipulation, which is equivalent to our given truth-value stipulation, can be given by using the axiom schemes A 1, B, A 5, A 6, and A 7 in the place of the axiom schemes A 1 to A 7 used in rule (1) for defining the axiomatic stipulation which has just been examined. If rules (2), (3), and (4) are left unaltered, then it is not difficult to show that the five axiom schemes A 1,

B, A 5, A 6, and A 7 can play the same role in rule (1) that the seven axiom schemes A 1 to A 7 have played in obtaining our above results. This is an immediate consequence of our remarks in chapter III concerning the relationship between an A-stipulation and a B-stipulation. In particular, observe that since $J_k(\ )$ and $\supset$ are now so interpreted that their corresponding truth functions satisfy standard conditions, an A-stipulation defined by A 1 to A 7 with R 1 is equivalent to a B-stipulation which is defined by A 1, B, A 5, A 6, and A 7 with R 1. Using this fact, one can readily check through the above results obtained by applying A 1 to A 7 and see that our still more elegant axiomatic stipulation using A 1, B, A 5, A 6, and A 7 has the desired properties.

Third illustration. In our first two illustrations, $b = 2$ and our choice of the functions F_1 and F_2 is such as to render our logical systems functionally incomplete. Let us now take $b = 3$ and choose F_1, F_2, and F_3 (C, N, and T) so that our logical systems are functionally complete. A moment's reflection will indicate that our new choice of functions (i.e., the addition of T) will not prevent us from obtaining a theorem 6. 3** for the present illustration when $S \geqslant 1$.

Likewise, if A 7 is altered properly to take account of our new choice for b and F_i's, then for $S = 1$ we can use the more elegant axiomatic stipulation of the first illustration (i.e., axioms defined by (1) and (2)) in order to get an axiomatic stipulation which is equivalent to the truth-value stipulation defined by our present choice for M, S, and the set of functions F_1, F_2, and F_3. In this connection it is helpful to note that both theorem 6. 6 and theorem 6. 8 remain valid with our addition of F_3. That is, the use which is made of substitution and $\theta(P)$ in obtaining our results is inde-, pendent of the structure determined by $F_3(P)$ (or TP). Similarly for $S > 1$ one can use (1) to (4) in either of the two forms given in the second illustration to obtain an axiomatic stipulation equivalent to our truth-value stipulation if A 7 is suitably altered.

Note that the change in A 7 which is required above for $S \geqslant 1$ amounts to an addition of M axiom schemes. Actually, a still more elegant axiomatic stipulation for $S \geqslant 1$ can be obtained by proceeding as has been indicated above except that in the place of the M axiom schemes added by the suitable changes in A 7, we add

the single axiom scheme $J_2(F_3(P))$. Clearly, this will always take a designated truth value, and since we have A 1 and R 1 it is easy to deduce the M axiom schemes which were used previously as a result of altering A 7. Hence, we can argue as before and establish the equivalence of our axiomatic and truth-value stipulations.

Fourth illustration. In this example, which is our last, we shall consider a case where $M = 4$, $S = 2$, $b = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 1$, $c = 1$, $\beta_1 = 1$, and $\gamma_1 = 1$. For this purpose, we will take &, $\sim$, and $\Diamond$ as in theorem 3. 11, and $\prod_1(X_1, P_1)$ as in the first illustration. If we now define $\supset$ and $J_k(\ )$ as in the proof of theorem 3. 11, one can easily obtain functions $\supset(p, q)$, $j_k(p)$, $\sim(p)$, and $N_r((X)P)$ which satisfy standard conditions. This follows by theorem 3. 11, the matrix (***m*** 1), and lemma 6. 2. 1*. Hence, it is not difficult to prove a theorem 6. 3*** for our present illustration. As in the case of the previous illustrations, there are probably more elegant axiomatic stipulations than the one indicated in theorem 6. 3***, but we shall leave it to the reader to discover such stipulations.

A little reflection will indicate that the proposed technique for establishing 6. 3*** makes it possible to introduce quantifiers into certain systems of strict implication. However, our method of formalization is in sharp contrast with the usual methods of formalizing systems of strict implication. This is apparent from our remarks on strict implication at the end of chapter III. In particular, our method guarantees deductive completeness with respect to a given truth-value stipulation, while this is not the case in standard treatments of strict implication. For example, in Barcan's "Functional calculus for strict implication" [48], the statement $(X) \sim \Diamond (P \,\&\, \sim(\sim\Diamond \sim\Diamond P))$ is not acceptable, while it is acceptable according to the axiomatic stipulation of the present illustration. To see that it is acceptable by our stipulation, note that $\sim\Diamond (P \,\&\, \sim(\sim\Diamond \sim\Diamond P))$ always takes the truth value 1 and that we have R 2. On the other hand, if the statement in question is acceptable in Barcan's system, then so is the "Brouwersche axiom" [49], but it is not difficult to show that this is impossible.

[48] Barcan 1946.

[49] See Lewis and Langford 1932, pp. 497–498.

VII

SOME UNSOLVED PROBLEMS

In the present work, a rather thorough treatment has been given of many-valued statement and predicate calculi. We have seen that these calculi may be thought of as generalizations of the ordinary two-valued statement and predicate calculus. The fact that it is thus possible to generalize the ordinary two-valued logic so as not only to cover the case of many-valued statement calculi, but of many-valued quantification theory as well, naturally suggests the possibility of further extending our treatment of many-valued logic to cover the case of many-valued sets, equality, numbers, etc. Since we now have a general theory of many-valued predicate calculi, there is little doubt about the possibility of successfully developing such extended many-valued theories. However, it is beyond the scope of the present work to consider the details of such theories, and we shall consider their careful study one of the major unsolved problems of many-valued logic. Hence, as yet, there is no answer to the following question:

I. If there are many-valued theories beyond the level of the predicate calculus, then what are the details of such theories?

Regardless of the ultimate answer to question I, it is of interest to raise questions concerning the further refinement of theories which are already developed. Under this category fall such questions as the simplification of hypotheses, axioms, proofs, etc. As far as the present work is concerned, there is one such question which we would like to mention in particular. This concerns the use which has been made of standard conditions in our development of quantification theory. It will be recalled that our proof in chapter V of the deductive completeness of many-valued predicate calculi requires a choice of functions $\supset(p, q)$, $j_k(p)$, $\sim(p)$, and $N_r((X)P)$ which satisfy standard conditions. However, there is no need of such an assumption of standard conditions in the proof of deductive completeness for many-valued statement calculi in chapter III. This state of affairs suggests our second question which is as follows:

II. Is it possible to develop a general theory of deductively complete many-valued predicate calculi without an assumption of standard conditions?

In ordinary two-valued logic, decision problems are of great theoretical interest, and such problems will continue to be of interest in the many-valued case. However, with the introduction of such concepts as F-decidability in the many-valued case, perhaps one would expect to encounter greater complexity in dealing with many-valued decision problems than in the two-valued case. This has already been suggested by the remarks of the previous chapters, and a closely related problem is raised by the following question:

III. Given an S and T such that $1 \leqslant S < T < M$, can one define a many-valued logic so that if P is a statement formula with the truth-value function p, then the following conditions are satisfied?

1. If p is such that we always have $p \leqslant S$, then $\vdash P$.
2. If p is ever such that $p > T$, then not-$\vdash P$.
3. If neither condition 1 nor condition 2 is satisfied, then we have $\vdash P$ in some cases and not-$\vdash P$ in other cases.

Granting the fact that there is considerable theoretical interest inherent in the possibility of developing a general system of many-valued logic through at least the level of the predicate calculus, from a pragmatic point of view it is natural to inquire concerning the possibility of useful applications of many-valued logical systems. This is the import of our next question.

IV. Are there useful applications of many-valued logics?

Thus far, attention has been concentrated on the possibility of so generalizing ordinary two-valued logic as to cover the case of many truth-values. In effect, the present work establishes the possibility of such a generalization through the level of the predicate calculus. However, are we to conclude from this that the significant consequences of such a generalization demand the use of the concept of many truth values? That is, could the same kind of generalization be achieved in effect without the introduction of the notion of many-valuedness? Answers to questions such as these can probably be inferred from the answer to the following question:

V. Precisely what problems (if any) can be solved by means of

many-valued logics ($M > 2$) which can not be solved by the ordinary two-valued logic?

The questions I to V are not meant to be exhaustive, but they do point to some major unsolved problems in the field of many-valued logic to which the present work gives little more than a hint concerning solutions. Also, the correct answers of questions I to V will determine in large measure the ultimate significance of the study of many-valued logics. This alone is sufficient to justify the study of such logics at least until the answers to our questions I to V have been found.

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GLOSSARY OF SYMBOLS

In this glossary we list the most important symbols which have been used in the present work. The symbols are listed in the order in which they are first introduced in the text, and the page numbers which follow each symbol refer either to that place at which the symbol is defined, or its properties discussed, or places where particularly significant uses are made of the given symbol.

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a_i	11	I	19
$F_i(P_1, \ldots, P_{a_i})$	11	*Lemma k.l.m*	19
S	12	$-(p)$	22
$f_i(p_1, \ldots, p_{a_i})$	12	$F_3(\ )$	23, 107
$p_1, p_2, \ldots, q, \ldots$	12	$t(p_1)$	23
$\supset$	14, 17, 22, 63, 103	$f_3(\ )$	23
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$K, A, C,$	15	$\Phi_w(P)$	23
$\max(p_1, p_2)$	15	$\phi_w(p)$	23
$\min(p_1, p_2)$	15	$\Psi_g(P)$	23
$\max(1, p_2 - p_1 + 1)$	15	$\psi_g(p)$	23
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$\{\mathrm{A}\ 1\}^*$	78
Δ	79
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$\{\ \}$	81
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$f_1^k(p, q)$	94
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ψ	97
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GENERAL INDEX